# Advanced Quantum Physics, Special Exercise

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1. **The Casimir Effect**
   Owing to quantum fluctuations of the electromagnetic field, there is an attractive force between two parallel metalic plates separated by a distance $d$, even if the two plates are located in a vaccum and are electrically neutral. This is known as the *Casimir effect*. For two plates of area $A$ seperated by a distance $d$, we have a shift of energy, which we will prove later, being

$$U(d,A) = -\frac{\pi^2}{720}\frac{\hbar A}{d^3}. \tag{1}$$

   It results that the force between the two plates is non zero and attractive

$$F = -\frac{\partial U(d,A)}{\partial d} = -\frac{\pi^2}{240}\frac{\hbar A}{d^4}\,. \tag{2}$$

   This has been confirmed experimentally in 1958 by Sparnaay (It was realized using $1cm^2$ Chrome-Steal plates ; at $d = 0.5\mu$ the attraction was $0.2dyn/cm^2$).

2. **Preparations**
   We consider the electromagnetic oscillations between two plates in a rectangular cavity (of dimensions $L_1 \times L_2 \times L_3$) with conducting walls. We must have $\mathbf{E}$ perpendicular and $\mathbf{B}$ tangential ( the transverse component of the electric field vanishes at the surface of a perfect conductor). These boundary conditions are satisified for a plane wave mode ($\sim e^{-i\omega t}$) if the components of the electric field have the following form

$$E_1 = E_1^0 \cos k_1x_1 \sin k_2x_2 \sin k_3x_3 e^{-i\omega t} \tag{3}$$
$$E_2 = E_2^0 \sin k_1x_1 \cos k_2x_2 \sin k_3x_3 e^{-i\omega t} \tag{4}$$
$$E_3 = E_3^0 \sin k_1x_1 \sin k_2x_2 \cos k_3x_3 e^{-i\omega t}\,, \tag{5}$$

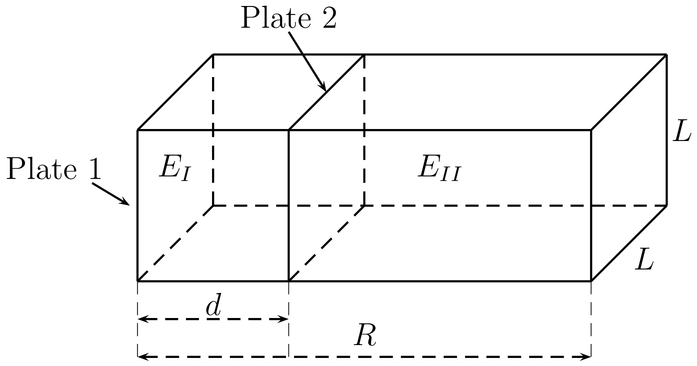
   where $k_i = n_i\pi/L_i$, $n_i$ an integer. It results that the possible frequencies $\omega$ for the wave modes must be restricted by the following dispersion relation

$$\frac{1}{c^2}\omega^2(n_1,n_2,n_3) = \mathbf{k}^2 = \pi^2\sum_i (n_i^2/L_i^2). \tag{6}$$

   The magnetic field $\mathbf{B}$ is obtain by the induction law $\nabla \times \mathbf{E} = i\frac{\omega}{c}\mathbf{B}$ and the corresponding boundary conditions are fullfilled automatically.
   The amplitudes $E_i^0$ are not arbitrary, since $\nabla \cdot \mathbf{E} = 0$, *ie* we must have

$$\sum_i E_i^0 k_i = 0. \tag{7}$$

   In general, equation( 7) has two linearly independent solutions (2 polarizations states). Excluded the case where the $n_i$ vanish. If there is only one of the $n_i$ being




zero, there is only one possible polarization mode. But if more $n_i$ vanish then there is no solution.

We consider the following setup.

Both plates are contained in a parallelogram with conducting walls. And the plates are parallel to the squared $(L \times L)$ ends of the box. One conducting plate is chosen to be at the beginning of the box. Where the second place is chosen to be at a distance $d$. In a another step this distance will be changed to $R/\eta$ (*e.g* $\eta = 2$). Now we construct

$$U(d, L, R) := E_I + E_{II} - (d \to R/\eta)\,, \tag{8}$$

where $E_I$, $E_{II}$ are the zeropoint energy of the free electromagnetic field within the subspaces of the box (see figure). The subspace with energy $E_{II}$ is only a tool to avoid divergence. We are interested in

$$U(d, L) = \lim_{R \to \infty} U(d, L, R)\,. \tag{9}$$

But each term in equation (8) are diverging separatly. Therefore we do a regularization of the sums for the zeropoint energy. Afterwards we calculate equations (8) and (9) and then undo the regularisations. A convenient regularization method is the following :

$$E_{I,II} \to E_{I,II}^{reg} = \sum_{\omega} \frac{1}{2}\hbar\omega e^{-\frac{\alpha}{\pi}\frac{\omega}{c}}\,. \tag{10}$$

**3. The steps for the regularization**

With equations (10) and (6) we have

$$E_I^{reg} = \frac{1}{2}\hbar c 2 \sum_{l,m,n} k_{l,m,n}(d, L, L) e^{-\frac{\alpha}{\pi} k_{l,m,n}(d,L,L)}\,, \tag{11}$$

where

$$k_{l,m,n}(d, L, L) = [(\frac{l\pi}{d})^2 + (\frac{m\pi}{L})^2 + (\frac{n\pi}{L})^2]^{1/2}\,. \tag{12}$$

Hence

$$U^{reg}(d, L, R, \alpha) = E_I^{reg} + (d \to R - d) - \{(d \to \frac{R}{\eta}) + (d \to R - \frac{R}{\eta})\}\,. \tag{13}$$

Now we consider the sum in equation (11). For very large $L$ we can replace the sums over $m$ and $n$ by integrals. (A more precise way is to study $U^{reg}(d,R,L^2,\alpha)/L^2$ when $L$ goes to infinity). We obtain then

$$E_I^{reg}(d,L,\alpha) = \hbar c \sum_{l=0}^{\infty} \int_0^{\infty} dm \int_0^{\infty} dn \sqrt{(\frac{l\pi}{d})^2 + (\frac{m\pi}{L})^2 + (\frac{n\pi}{L})^2} e^{-\frac{\alpha}{\pi}\sqrt{(\frac{l\pi}{d})^2+(\frac{m\pi}{L})^2+(\frac{n\pi}{L})^2}} . \tag{14}$$

In equation (13) the term with $l = 0$ does not contribute to the sum. Therefore we can neglect it in $E_I^{reg}$. At first transform equation (14) into

$$E_I^{reg} = -\frac{\pi^2}{4}\hbar c L^2 \frac{d^3}{d\alpha^3} \sum_{l=1}^{\infty} \int_0^{\infty} \frac{dz}{1+z} e^{-\frac{l}{d}\alpha\sqrt{1+z}} . \tag{15}$$

Afterwards perform the sum over $l$ and then take the derivative with respect to $\alpha$ and bring this result to the form

$$E_I^{reg} = \frac{\pi^2 \hbar c L^2}{2d} \frac{d^2}{d\alpha^2} \frac{d/\alpha}{e^{\alpha/d} - 1} . \tag{16}$$

Hint :

$$\frac{y}{e^y - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} y^n , \tag{17}$$

where the $B_n$ are the Bernoulli numbers and show

$$U(d,L) := \lim_{R\to\infty} \lim_{\alpha\to 0} U^{reg}(d,L,R,\alpha) = -\frac{\pi^2}{720} \frac{\hbar c L^2}{d^3}, \tag{18}$$

which corresponds to equation (1).