Advanced Quantum Physics, Exercises X

Prof. Hans Peter Büchler SS 2011, 21st of December 2011

1. Rabi Oscillations

We consider a Spin$-1/2$ particle with the Hamiltonian

$$H = \frac{\hbar\Omega}{2}\Big[|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\Big] = \frac{1}{2}\begin{pmatrix} 0 & \hbar\Omega \\ \hbar\Omega & 0 \end{pmatrix} \tag{1}$$

At time $t = 0$ we prepare the system in the state $|\uparrow\rangle$.

(a) Compute the time evolution of the system with the Hamiltonian H.

(b) Calculate the expectations values $\langle\sigma_z\rangle$ and $\langle\sigma_x\rangle$, and draw the solutions as a function of time t.

(c) Find the probability to be in the state $|\uparrow\rangle$ or in state $|\downarrow\rangle$ at time $t_1 = 3/(2\Omega)$.

(d) A measurement at time t_1 found the spin to be in the state $|\uparrow\rangle$. In which state is the spin at time $t_2 = 7/(2\Omega)$?

2. 1 D System

Consider a particle in the following 1-D Potential

$$V(x) = \begin{cases} \infty & x < -a \\ V_0\delta(x) & -a < x < a \\ \infty & x > a \end{cases} \tag{2}$$

(a) Write the Hamiltonian of the system. Which commutation relations x and p fulfill?

(b) Formulate the boundary conditions of the wave function at the points $x = -a$, $x = a$, and $x = 0$.

(c) Show that the asymmetric wave functions $\psi(x) = \sin\left(\frac{\pi x n}{a}\right)$ with n (an integer) are eigenstates of the Hamiltonian. What are the associated eigenenergies?

(d) Find the symmetric solutions and determine the ground state energy.

3. Harmonic oscillator

Consider the 1-D harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2. \tag{3}$$

(a) Give the expression of the operators $a^\dagger$ and a in terms of x and p. What is the Hamiltonian form with those operators?

(b) Determine the following commutation relations

$$[a, a^\dagger] \qquad [x, a^\dagger] \qquad [(a)^n, a^\dagger] \qquad [H, x] \qquad [H, p] \tag{4}$$

with n an integer

(c) Write the form of those operators in Heisenberg picture

$$a_{\mathrm{H}}(t) \qquad a^\dagger_{\mathrm{H}}(t) \qquad x_{\mathrm{H}}(t) \qquad p_{\mathrm{H}}(t). \tag{5}$$

(d) Find the ground state wave function $\psi_0(x)$. Calculate the wave function $\psi_1(x)$ by applying the appropriate operator.

4. Perturbation theory

A hydrogen atom is disturbed by an harmonic oscillator along the z-axis

$$V_{\mathrm{ext}}(\mathbf{x}) = kz^2. \tag{6}$$

(a) Compute the energy correction of the ground state in first order of perturbation theory.

(b) Consider now the excited state with principal quantum number $n = 2$. How big is the degeneracy ? What are the other quantum number characterising this state.

(c) The perturbation now leads to a splitting of these energy degeneracy. By symmetry we can say how this splitting of the degeneracy appears. What are the good quantum numbers ? (Arguments) What is the state with the lowest energy, and with the highest energy ?

(d) Check your arguments in (c) doing the explicit calculation.

5. Hilbert space

We consider a particle with orbital angular momentum $l = 2$ (d-wave) and intrinsic spin $S = 3/2$.

(a) What is the dimension of the Hilbert space and give the natural basis of eigenstates.

(b) What are the allowed values for the total angular momentum J ?

(c) What is the parity of the eigenstates ?

(d) We give the spin-orbit coupling Hamiltonian $H = \gamma \mathbf{L} \cdot \mathbf{S}$. Calculate the eigenenergies of the system and give the degeneracy.