1. Rabi Oscillations

We consider a Spin–1/2 particle with the Hamiltonian

\[ H = \frac{\hbar \Omega}{2} \left[ |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| \right] = \frac{1}{2} \begin{pmatrix} 0 & \hbar \Omega \\ \hbar \Omega & 0 \end{pmatrix} \]  

At time \( t = 0 \) we prepare the system in the state \( |\uparrow\rangle \).

(a) Compute the time evolution of the system with the Hamiltonian \( H \).

(b) Calculate the expectations values \( \langle \sigma_z \rangle \) and \( \langle \sigma_x \rangle \), and draw the solutions as a function of time \( t \).

(c) Find the probability to be in the state \( |\uparrow\rangle \) or in state \( |\downarrow\rangle \) at time \( t_1 = 3/(2\Omega) \).

(d) A measurement at time \( t_1 \) found the spin to be in the state \( |\uparrow\rangle \). In which state is the spin at time \( t_2 = 7/(2\Omega) \) ?

2. 1 D System

Consider a particle in the following 1-D Potential

\[ V(x) = \begin{cases} \infty & x < -a \\ V_0\delta(x) & -a < x < a \\ \infty & x > a \end{cases} \]  

(a) Write the Hamiltonian of the system. Which commutation relations \( x \) and \( p \) fulfill?

(b) Formulate the boundary conditions of the wave function at the points \( x = -a \), \( x = a \), and \( x = 0 \).

(c) Show that the asymmetric wave functions \( \psi(x) = \sin \left( \frac{2\pi n}{a} \right) \) with \( n \) (an integer) are eigenstates of the Hamiltonian. What are the associated eigenenergies?

(d) Find the symmetric solutions and determine the ground state energy.

3. Harmonic oscillator

Consider the 1-D harmonic oscillator

\[ H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2. \]  

(a) Give the expression of the operators \( a^\dagger \) and \( a \) in terms of \( x \) and \( p \). What is the Hamiltonian form with those operators?
(b) Determine the following commutation relations
\[
\begin{align*}
[a, a^\dagger] & \quad [x, a^\dagger] & \quad [(a)^n, a^\dagger] & \quad [H, x] & \quad [H, p] \\
\end{align*}
\] (4)

with \( n \) an integer

(c) Write the form of those operators in Heisenberg picture
\[
a_H(t) \quad a_H^\dagger(t) \quad x_H(t) \quad p_H(t).
\] (5)

(d) Find the ground state wave function \( \psi_0(x) \). Calculate the wave function \( \psi_1(x) \) by applying the appropriate operator.

4. Perturbation theory

A hydrogen atom is disturbed by an harmonic oscillator along the \( z \)-axis
\[
V_{\text{ext}}(x) = k z^2.
\] (6)

(a) Compute the energy correction of the ground state in first order of perturbation theory.

(b) Consider now the excited state with principal quantum number \( n = 2 \). How big is the degeneracy? What are the other quantum number characterising this state.

(c) The perturbation now leads to a splitting of these energy degeneracy. By symmetry we can say how this splitting of the degeneracy appears. What are the good quantum numbers? (Arguments) What is the state with the lowest energy, and with the highest energy?

(d) Check your arguments in (c) doing the explicit calculation.

5. Hilbert space

We consider a particle with orbital angular momentum \( l = 2 \) (d-wave) and intrinsic spin \( S = 3/2 \).

(a) What is the dimension of the Hilbert space and give the natural basis of eigenstates.

(b) What are the allowed values for the total angular momentum \( J \)?

(c) What is the parity of the eigenstates?

(d) We give the spin-orbit coupling Hamiltonian \( H = \gamma \mathbf{L} \cdot \mathbf{S} \). Calculate the eigenenergies of the system and give the degeneracy.