Advanced Quantum Physics, Exercises II

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1. Harmonic Oscillator

We consider a one-dimensional harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2. \tag{1}$$

(a) Write the definition of the creation operator $a^\dagger$ and the annihilation operator a in terms of the position operator x and the momentum operator p. What is the form of the hamiltonian using those new operators ?

(b) Calcultate the following commutation relation:

$$[a, a^\dagger] \qquad [x, a^\dagger] \qquad [(a)^n, a^\dagger] \qquad [H, x] \qquad [H, p] \tag{2}$$

with n being a natural integer.

(c) We continue now with the Heisenberg picture. Estimate those operators

$$a_{\mathrm{H}}(t) \qquad a^\dagger_{\mathrm{H}}(t) \qquad x_{\mathrm{H}}(t) \qquad p_{\mathrm{H}}(t) \tag{3}$$

in Heisenberg picture.

(d) Find the groundstate wave function $\psi_0(x)$. Calculate the wave function $\psi_1(x)$ of the first excited state by applying the creation operator.

2. Heisenberg Uncertainity Relation

(a) Show, for the momentum and position operators p and x, this uncertainty relation

$$\overline{\Delta p} \cdot \overline{\Delta x} \geq \frac{\hbar}{2}.$$

Where for an operator A we define the variance as $\overline{\Delta A} = \sqrt{\langle (\Delta A)^2 \rangle}$ and the deviation from the expectation value as $\Delta A = A - \langle A \rangle$.

(b) Given two hermitian operators A and B, prove that the generalized uncertainty principle

$$\Delta^2(A) \cdot \Delta^2(B) \geq \frac{\langle i[A, B] \rangle^2}{4}$$

is fulfilled, where $\langle A \rangle$ is the expectation value and $\Delta^2(A) = \langle (A - \langle A \rangle)^2 \rangle$ the standard deviation of A in a given state. Then show that this result is consistent with part (a).