1. **Harmonic Oscillator**

We consider a one-dimensional harmonic oscillator

\[ H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2. \]  

(a) Write the definition of the creation operator \( a^\dagger \) and the annihilation operator \( a \) in terms of the position operator \( x \) and the momentum operator \( p \). What is the form of the hamiltonian using those new operators?

(b) Calculate the following commutation relation:

\[ [a, a^\dagger] \quad [x, a^\dagger] \quad [(a)^n, a^\dagger] \quad [H, x] \quad [H, p] \]  

with \( n \) being a natural integer.

(c) We continue now with the Heisenberg picture. Estimate those operators

\[ a_H(t) \quad a_H^\dagger(t) \quad x_H(t) \quad p_H(t) \]  

in Heisenberg picture.

(d) Find the groundstate wave function \( \psi_0(x) \). Calculate the wave function \( \psi_1(x) \) of the first excited state by applying the creation operator.

2. **Heisenberg Uncertainty Relation**

(a) Show, for the momentum and position operators \( p \) and \( x \), this uncertainty relation

\[ \Delta p \cdot \Delta x \geq \frac{\hbar}{2}. \]

Where for an operator \( A \) we define the variance as \( \Delta A = \sqrt{\langle (\Delta A)^2 \rangle} \) and the deviation from the expectation value as \( \Delta A = A - \langle A \rangle \).

(b) Given two hermitian operators \( A \) and \( B \), prove that the generalized uncertainty principle

\[ \Delta^2(A) \cdot \Delta^2(B) \geq \frac{|i[A, B]|^2}{4} \]

is fulfilled, where \( \langle A \rangle \) is the expectation value and \( \Delta^2(A) = \langle (A - \langle A \rangle)^2 \rangle \) the standard deviation of \( A \) in a given state. Then show that this result is consistent with part (a).