1. Plane Waves (Homework)

a) Plane waves in the interval \([0, L]\) with periodic boundary conditions are given by

\[
\psi_n = \frac{1}{\sqrt{L}} \exp \left(\frac{i}{\hbar} p_n x\right) \quad \text{with} \quad p_n = \frac{2\pi \hbar n}{L}.
\]  

(1)

Show that it forms a complete set of orthonormal functions. To do so, show the validity of the following relations

\[
\int_0^L dx \psi_n^*(x) \psi_m(x) = \delta_{n,m} : \text{Orthogonalization} \quad (2)
\]

\[
\sum_{n=-\infty}^{\infty} \psi_n^*(x) \psi_n(x') = \delta(x-x') : \text{Completeness} \quad (3)
\]

b) In the limit \(L \to \infty\) the two previous relations (2) and (3) take the form

\[
\int dx \exp \left[-\frac{i}{\hbar} (p-p') x\right] = 2\pi \hbar \delta(p-p') : \text{Orthogonalization} \quad (4)
\]

\[
\int \frac{dp}{2\pi \hbar} \exp \left[\frac{i}{\hbar} (x-x') p\right] = \delta(x-x') : \text{Completeness} \quad (5)
\]

Take explicitly the limit \(L \to \infty\) for \(\psi_n\) using a suitable prefactor and show those relations for orthogonalization and completeness (4) and (5). (The basis functions are normalized and are therefore called a generalized set of basis functions.)

2. Commutation relations (Oral)

a) Show the identity \([AB, C] = A[B, C] + [A, C] B\).

b) Compute \([x, p^2]\), \([x^2, p^2]\) and \([xp, p^2]\).

c) We now consider \(g(x), f(p)\) with an existent Taylor expansion. Show, that \([p, g(x)] = -i\hbar \frac{dg}{dx} g(x)\) and \([x, f(p)] = i\hbar \frac{df}{dp} f(p)\).

3. Baker-Campell-Hausdorff-Formulae (Homework)

We consider two noncommutative operators \(A\) and \(B\), with the following relation \([A, [A, B]] = [B, [A, B]] = 0\).

(a) Show, that for the operators \(A\) and \(B\) the relation

\[
e^{-At} Be^{At} = B - t[A, B]
\]  

is fulfilled, for any number \(t\).
(b) Prove the Baker-Campell-Hausdorff-Formulae
\[ e^{A+B} = e^A e^B e^{-[A,B]/2} \tag{7} \]
for those operators. (Hint : Write a first order differential equation for the operator \( W(t) = e^{-tA} e^{t(A+B)} \) and solve it.)

4. Density matrix (Homework)

(a) A beam of Spin-1 particles passes through the modified Stern-Gerlach-Apparatus (\( \alpha \)) and (\( \beta \)). The spin state of the beam is given by the density matrix \( \rho_0 = \frac{1}{3} |1\rangle \langle 1| \).
What is the density operator \( \rho_\alpha \) and \( \rho_\beta \) for the beam after it passed through the apparatus (\( \alpha \)) or (\( \beta \))? Show, that for those density operators, we have \( \text{Tr} \rho_\alpha/\beta = 1 \) and \( \text{Tr} \rho^2_\alpha/\beta \leq 1 \). What is the physical difference between \( \text{Tr} \rho^2_\alpha/\beta = 1 \) and \( \text{Tr} \rho^2_\alpha/\beta < 1 \)?
General hint : beware of normalization!

![Figure 1 – Stern-Gerlach-Apparatus (\( \alpha \)) and (\( \beta \))]  

(b) Now we consider a beam of Spin-1/2 particles whose state is given by the following density operator
\[ \rho_0 = |+\rangle \frac{1}{4} \langle +| + |-\rangle \frac{3}{4} \langle -|. \tag{8} \]
This beam passes through the Stern-Gerlach-Apparatus (\( \gamma \)). What is the density operator \( \rho_\gamma \) for the beam after it has passed through the apparatus? Does the outgoing beam represent a pure state?

![Figure 2 – Stern-Gerlach-Apparatus (\( \gamma \))]  

(c) Construct a density matrix for a Spin-1/2 beam which is fully polarized in the positive \( x \) direction, using the basis \( \{|s_z,+\rangle, |s_z,-\rangle\} \). How large is the expectation value for the polarization of the spins in the \( y \) direction?

(d) Construct a density matrix for a Spin-1/2 beam mixture, where 3/4 of the particles are polarized in the positive \( z \) direction and 1/4 are polarized in the positive \( x \) direction, in the basis \( \{|s_z,+\rangle, |s_z,-\rangle\} \). Compute the expectation values of \( s_x \), \( s_y \) and \( s_z \).