Advanced Quantum Physics, Exercises III

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1. Plane Waves (Homework)

a) Plane waves in the interval $[0,\, L]$ with periodic boundary conditions are given by

$$\psi_n = \frac{1}{\sqrt{L}} \exp\left(\frac{i}{\hbar}\, p_n\, x\right) \qquad \text{with} \qquad p_n = \frac{2\pi\hbar n}{L}\,. \tag{1}$$

Show that it forms a complete set of orthonormal functions. To do so, show the validity of the following relations

$$\int_0^L dx\, \left[\psi_n(x)\right]^* \psi_m(x) = \delta_{n,m} \qquad : \text{Orthogonalization} \tag{2}$$

$$\sum_{n=-\infty}^{\infty} \left[\psi_n(x)\right]^* \psi_n(x') = \delta(x - x') : \text{Completeness} \tag{3}$$

b) In the limit $L \to \infty$ the two previous relations (2) and (3) take the form

$$\int dx\, \exp\left[-\frac{i}{\hbar}\,(p - p')\; x\right] = 2\pi\hbar\; \delta(p - p') : \text{Orthogonalization} \tag{4}$$

$$\int \frac{dp}{2\pi\hbar} \exp\left[\frac{i}{\hbar}\,(x - x')\; p\right] = \qquad \delta(x - x') : \text{Completeness} \tag{5}$$

Take explicity the limit $L \to \infty$ for ψ_n using a suitable prefactor and show those relations for orthogonalization and completeness (4) and (5). (The basis functions are normalized and are therefore called a generalized set of basis functions.)

2. Commutation relations (Oral)

a) Show the identity $[AB, C] = A\,[B, C] + [A, C]\,B$.

b) Compute $[x, p^2]$, $[x^2, p^2]$ and $[xp, p^2]$.

c) We now consider $g(x)$, $f(p)$ with an existent Taylor expansion. Show, that $[p, g(x)] = -i\hbar\, \frac{d}{dx}\, g(x)$ and $[x, f(p)] = i\hbar\, \frac{d}{dp}\, f(p)$.

3. Baker-Campell-Hausdorff-Formulae (Homework)

We consider two noncommutative operators A and B, with the following relation $[A, [A, B]] = [B, [A, B]] = 0$.

(a) Show, that for the operators A and B the relation

$$\mathrm{e}^{-At} B \mathrm{e}^{At} = B - t[A, B] \tag{6}$$

is fulfilled, for any number t.

(b) Prove the Baker-Campell-Hausdorff-Formulae

$$e^{A+B} = e^{A}e^{B}e^{-[A,B]/2} \tag{7}$$

for those operators. (Hint : Write a first order differential equation for the operator $W(t) = e^{-tA}e^{t(A+B)}$ and solve it.)

4. Density matrix (Homework)

(a) A beam of Spin-1 particles passes through the modified Stern-Gerlach-Apparatus (α) and (β). The spin state of the beam is given by the density matrix $\rho_0 = \frac{1}{3}\mathbb{1}$. What is the density operator ρ_α and ρ_β for the beam after it passed through the apparatus (α) or (β) ? Show, that for those density operators, we have $\mathrm{Tr}\,\rho_{\alpha/\beta} = 1$ and $\mathrm{Tr}\,\rho^2_{\alpha/\beta} \leq 1$. What is the physical difference between $\mathrm{Tr}\,\rho^2_{\alpha/\beta} = 1$ and $\mathrm{Tr}\,\rho^2_{\alpha/\beta} < 1$?
General hint : beware of normalization !

FIGURE 1 – Stern-Gerlach-Apparatus (α) and (β)

(b) Now we consider a beam of Spin-1/2 particles whose state is given by the following density operator

$$\rho_0 = |+\rangle\frac{1}{4}\langle+| + |-\rangle\frac{3}{4}\langle-|. \tag{8}$$

This beam passes through the Stern-Gerlach-Apparatus (γ). What is the density operator ρ_γ for the beam after it has passed through the apparatus ? Does the outgoing bam represent a pure state ?

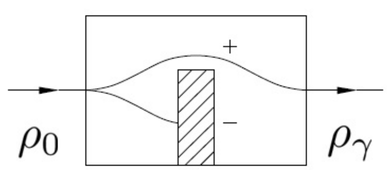

FIGURE 2 – Stern-Gerlach-Apparatus (γ)

(c) Construct a density matrix for a Spin-1/2 beam which is fully polarized in the positive x direction, using the basis $\{|s_z,+\rangle, |s_z,-\rangle\}$. How large is the expectation value for the polarization of the spins in the y direction ?

(d) Construct a density matrix for a Spin-1/2 beam mixture, where 3/4 of the particles are polarized in the positive z direction and 1/4 are polarized in the positive x direction, in the basis $\{|s_z,+\rangle, |s_z,-\rangle\}$. Compute the expectation values of s_x, s_y and s_z.