Advanced Quantum Physics, Exercises IV

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1. Harmonic Oscillator in the Heisenberg picture (Oral)

The operators in the Heisenberg picture are linked to the Schrödinger picture operators via the relation

$$A_H(t) = U^{-1}(t) A_S U(t), \qquad U(t) = e^{-\frac{i}{\hbar}Ht}. \tag{1}$$

Where $U(t)$ is the time evolution operator, and fulfills the *Schrödinger equation* $i\hbar\partial_t U(t) = HU(t)$. We define the states in Heisenberg picture as $|\psi_H\rangle := |\psi_S(t=0)\rangle$. The subscript H stands for *Heisenberg*, S for *Schrödinger*. In this context the operators $A_H(t)$ and A_S are arbitrary operators in Heisenberg and Schrödinger picture respectively.

(a) Derive from eq. (1) the Heisenberg equation of motion for the operators

$$\partial_t A_H(t) = \frac{i}{\hbar}[H, A_H(t)]. \tag{2}$$

(b) We now consider the Hamiltonian for the Harmonic Oscillator

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2 = \hbar\omega\left(a_S^\dagger a_S + \frac{1}{2}\right), \qquad [a_S, a_S^\dagger] = 1. \tag{3}$$

Show

$$a_H(t) = e^{-i\omega t} a_S, \qquad a_H^\dagger(t) = e^{+i\omega t} a_S^\dagger. \tag{4}$$

Where the operators a_S and $a_S^\dagger$ are the annihilation and creation operators of the Harmonic Oscillator in Schrödinger picture.

(c) Prove the following relations

$$Q_H(t) = Q_H(t; P_S, Q_S), \qquad P_H(t) = P_H(t; P_S, Q_S). \tag{5}$$

Hint : $0 = e^{-i\omega t} a_S - e^{-i\omega t} a_S$ and $0 = e^{i\omega t} a_S^\dagger - e^{i\omega t} a_S^\dagger$.

(d) Show, that the Heisenberg equation of motion (eq 2) leads to the classical Hamilton equations for the operators $Q_H(t)$ and $P_H(t)$ for the harmonic oscillator.

(e) We define at time $t = 0$ the state $|\psi_S(t=0)\rangle = |\psi_H\rangle = |1\rangle + |2\rangle =: |\gamma\rangle$ ($N|n\rangle = n|n\rangle$). Calculate the expectation values

$$\langle Q_H(t)\rangle_\gamma = \langle\gamma|Q_H(t)|\gamma\rangle, \qquad \langle P_H(t)\rangle_\gamma = \langle\gamma|P_H(t)|\gamma\rangle, \tag{6}$$

for arbitrary times t.

2. Free particles in Heisenberg picture (Homework)

Consider the one dimensional Hamiltonian for free particles of mass m :

$$H = \frac{1}{2m} p^2. \tag{7}$$

(a) Solve the equation of motion for the position operator $q_H(t)$ and the momentum operator $p_H(t)$ in the Heisenberg picture.

(b) Calculate the following commutators :

$$[q_H(t_1), q_H(t_2)] \tag{8}$$
$$[p_H(t_1), p_H(t_2)] \tag{9}$$
$$[q_H(t_1), p_H(t_2)] \tag{10}$$