Advanced Quantum Physics, Exercises VIII

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1. Fermi's golden rule in a periodic potential (Homework)
Derive the Fermi's golden rule for the following periodic perturbation

$$H_1 = V\ \cos\omega t\, e^{\eta t}. \tag{1}$$

2. Time dependent perturbation theory (Homework)

We analyze a one-dimensional harmonic oscillator with mass m, frequency ω within a time dependent electric field. The Hamiltonian takes the form $H = H_0 + H'(t)$ with $H_0 = p^2/2m + m\omega^2 x^2/2$ the harmonic oscillator and the perturbation $H'(t) = xD(t)$. The time dependence of the external electric field takes the form

$$D(t) = A\frac{1}{\sqrt{\pi}\tau}e^{-(t/\tau)^2}. \tag{2}$$

(a) Determine within first order perturbation theory the probability for a transition from the ground state into an excited state $P_{0\to n}(\infty)$, i.e., the transition probability in the long time limit $t \to \infty$. What happens for $\tau \to 0$?

(b) The transition probability can also be solved exactly. Prove that the exact transition probability takes the form

$$P_{0\to n}(\infty) = \frac{K^{2n}}{n!}e^{-K^2} \qquad \text{with} \qquad K = \frac{Ae^{-\omega^2\tau^2/4}}{\sqrt{2m\omega\hbar}} \tag{3}$$

and compare with the perturbative result in (a).
Hints : Work in the Dirac picture and derive the Schrödinger equation

$$i\partial_t|\psi_D(t)\rangle = H_D(t)|\psi_D(t)\rangle \qquad \text{with} \qquad H_D = \sqrt{\frac{\hbar}{2m\omega}}D(t)\left[e^{-i\omega t}a + e^{i\omega t}a^\dagger\right]. \tag{4}$$

Make the ansatz $|\psi_D(t)\rangle = e^{-iK(t)a^\dagger}|\bar{\psi}(t)\rangle$ and choose $K(t)$ properly to eliminate $a^\dagger$ from the Schrödinger equation. Using the relation $e^{iKa^\dagger}ae^{-iKa^\dagger} = a - iK$ the remaining equation is trivial to integrate and the exact solution $|\psi(t)\rangle$ for the wave function in the Schrödinger picture follows. The transition probabilities then derive from $P_{0\to n}(\infty) = |\langle n|\psi(\infty)\rangle|^2$.

3. Hydrogen atom, Fermi's golden rule (Oral)

The ground state of a hydrogen atom ($n = 1\,, l = 0$) is exposed to a time-dependent potential. This perturbation has the form

$$V\left(\mathbf{x}, t\right) = V_0\,\cos\left(kz - \omega t\right)\,. \tag{5}$$

(a) Using time-dependent perturbation theory calculate the transition rate at which an electron with momentum $\mathbf{p}$ is emitted. We suppose here that the emitted electrons can be described by plane waves.

(b) How is the angular distribution of emitted electrons ?

Hint : We give the form of the ground state wave function of the hydrogen atom

$$\Psi_{n=1,l=0}(\mathbf{x}) = \frac{1}{\sqrt{\pi}} \left(\frac{1}{a_0} \right)^{3/2} e^{-r/a_0} ,$$

where a_0 is the Bohr radius.