Advanced Quantum Physics, Exercises IX

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1. Harmonic oscillator and coherent states (Homework)
For any $\alpha \in \mathbb{C}$ we define the coherent state

$$|\alpha\rangle \equiv \mathrm{e}^{-|\alpha|^2/2} \sum_{n\geq 0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,. \tag{1}$$

a) Show, that

$$|\alpha\rangle = \mathrm{e}^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}} |0\rangle \,, \tag{2}$$

and

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{3}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the harmonic oscillator annihilation and creation operators seen in class.(Hint : $\mathrm{e}^{-\bar{\alpha}\hat{a}}\,|0\rangle = |0\rangle$)

b) Calculate $\langle x\rangle_\alpha \equiv \langle\alpha|\,\hat{x}\,|\alpha\rangle$, $\langle p\rangle_\alpha$, Δx_α, Δp_α show, that for all $\alpha \in \mathbb{C}$,

$$\Delta x_\alpha \Delta p_\alpha = \frac{\hbar}{2}, \tag{4}$$

applies, i.e that coherent states minimize the uncertainty.
(Hint : $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$)

c) Show that coherent states are not orthogonal but we instead have the relation

$$\langle\alpha|\beta\rangle = \mathrm{e}^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2-2\bar{\alpha}\beta\right)}. \tag{5}$$

2. Quantized fields and coherent states (Homework)
The mode of the electric field with momentum $\mathbf{k}$ and polarization λ is occupied by a coherent state, i.e., $|\alpha_{\mathbf{k},\lambda}\rangle = e^{\alpha a^\dagger_{\mathbf{k},\lambda} - a_{\mathbf{k},\lambda}\alpha^*}|0\rangle$. Such a state describes a perfect laser.

a) Calculate the expectation values of $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$

b) Determine the measured field intensity defined as $\langle \mathbf{E}^-(\mathbf{x},t)\mathbf{E}^+(\mathbf{x},t)\rangle$ at a detector at position $\mathbf{x}$ and time t.

3. Single photon states (Oral)

We focus on a single photon state $|\mathbf{k},\lambda\rangle = a^\dagger_{\mathbf{k},\lambda}|0\rangle$.

a) Determine the expectation value of the single photon state with the electric $\mathbf{E}(\mathbf{x},t)$ and magnetic field $\mathbf{B}(\mathbf{x},t)$.

b) What are the quadratic fluctuations of the electric field and magnetic field at position $\mathbf{x}$ and time t ? Is the situation different for coherent states ?

c) What are the measured field value $\langle \mathbf{E}^-(\mathbf{x},t)\mathbf{E}^+(\mathbf{x},t)\rangle$ for the single photon state ?