Theoretische Physik: Fortgeschrittene Vielteilchentheorie
Weihnachtsaufgabe

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The Casimir Effect

Owing to quantum fluctuations of the electromagnetic field, there is an attractive force between two parallel metallic plates separated by a distance d, even if the two plates are located in a vacuum and are electrically neutral. This is known as the *Casimir effect.* For two plates of area A separated by a distance d, we have a shift of energy, which we will prove later, being

$$U(d,A) = -\frac{\pi^2}{720}\frac{\hbar A}{d^3}. \tag{1}$$

It results that the force between the two plates is non zero and attractive

$$F = -\frac{\partial U(d,A)}{\partial d} = -\frac{\pi^2}{240}\frac{\hbar A}{d^4}\,. \tag{2}$$

This has been confirmed experimentally in 1958 by Sparnaay (It was realized using $1cm^2$ Chrome-Steal plates; at $d = 0.5\mu$ the attraction was $0.2dyn/cm^2$).

1. Preparations

We consider the electromagnetic oscillations between two plates in a rectangular cavity (of dimensions $L_1 \times L_2 \times L_3$) with conducting walls. We must have $\mathbf{E}$ perpendicular and $\mathbf{B}$ tangential (the transverse component of the electric field vanishes at the surface of a perfect conductor).These boundary conditions are satisfied for a plane wave mode ($\sim e^{-i\omega t}$) if the components of the electric field have the following form

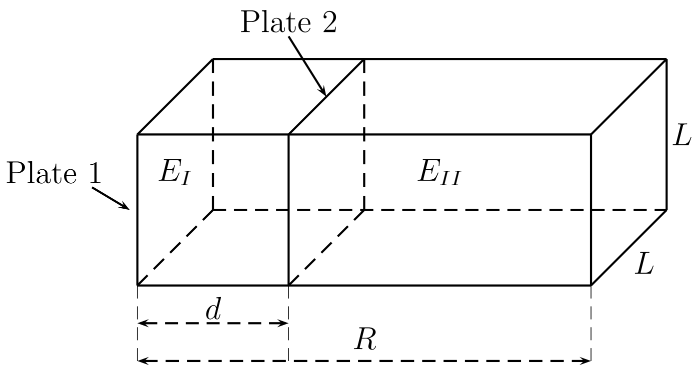
$$E_1 = E_1^0 \cos k_1x_1 \,\sin k_2x_2 \,\sin k_3x_3 \, e^{-i\omega t} \tag{3}$$
$$E_2 = E_2^0 \sin k_1x_1 \,\cos k_2x_2 \,\sin k_3x_3 \, e^{-i\omega t} \tag{4}$$
$$E_3 = E_3^0 \sin k_1x_1 \,\sin k_2x_2 \,\cos k_3x_3 \, e^{-i\omega t}\,, \tag{5}$$

where $k_i = n_i\pi/L_i$ and n_i being an integer. It results that the possible frequencies ω for the wave modes must be restricted by the following dispersion relation

$$\frac{1}{c^2}\omega^2(n_1,n_2,n_3) = \mathbf{k}^2 = \pi^2\sum_i (n_i^2/L_i^2). \tag{6}$$

The magnetic field $\mathbf{B}$ is obtain by the induction law $\nabla \times \mathbf{E} = i\,(\omega/c)\,\mathbf{B}$ and the corresponding boundary conditions are fulfilled automatically.
The amplitudes E_i^0 are not arbitrary, since $\nabla \cdot \mathbf{E} = 0$, *i.e.* we must have

$$\sum_i E_i^0 k_i = 0. \tag{7}$$



In general, equation (7) has two linearly independent solutions (2 polarizations states). Excluded the case where the n_i vanish. If there is only one of the n_i being zero, there is only one possible polarization mode. But if more n_i vanish then there is no solution.

We consider the following setup.

Both plates are contained in a parallelogram with conducting walls. The plates are parallel to the squared ($L \times L$) ends of the box. One conducting plate is chosen to be at the beginning of the box. Where the second plate is chosen to be at a distance d. In another step the distance will be changed to R/η ($e.g\ \eta = 2$). Now we construct

$$U(d, L, R) := E_I(d) + E_{II}(R - d) - (E_{III}(R/\eta) + E_{IV}(R - R/\eta)) \ , \tag{8}$$

where E_I, E_{II} are the zero point energy of the free electromagnetic field within the subspaces of the box (see figure). The subspace with energy E_{II} is only a tool to avoid divergence. We are interested in

$$U(d, L) = \lim_{R \to \infty} U(d, L, R) \ . \tag{9}$$

But each term in equation (8) are diverging separately. Therefore we do a regularization of the sums for the zero point energy. Afterwards we calculate equations (8) and (9) and then undo the regularisations. A convenient regularization method is the following

$$E_{I,II} \to E_{I,II}^{reg} = \sum_{\omega} \frac{1}{2} \hbar\omega \ \exp[-\alpha\omega/\pi c] \ . \tag{10}$$

2. The steps for the regularization

With equations (10) and (6) we have

$$E_I^{reg} = \hbar c \sum_{l,m,n} k_{l,m,n}(d, L, L) \ \exp[-(\alpha/\pi) \, k_{l,m,n}(d, L, L))] \, , \tag{11}$$

where

$$k_{l,m,n}(d, L, L) = \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \ . \tag{12}$$

Hence

$$U^{reg}\,(d,L,R,\alpha) = E_I^{reg}(d) + E_{II}^{reg}(R-d) - \{E_{III}^{reg}(R/\eta) + E_{IV}^{reg}(R-R/\eta)\}\ . \quad (13)$$

Now we consider the sum in equation (11). For very large L we can replace the sums over m and n by integrals. (A more precise way is to study $U^{reg}(d,R,L^2,\alpha)/L^2$ when L goes to infinity). We obtain then

$$\begin{aligned} E_I^{reg}(d,L,\alpha) &= \hbar c \sum_{l=0}^{\infty} \int_0^{\infty} dm \int_0^{\infty} dn \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \times \\ & \exp\left[-\frac{\alpha}{\pi}\sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2}\,\right]\ . \end{aligned} \quad (14)$$

In equation (13) the term with $l=0$ does not contribute to the sum. Therefore we can neglect it in E_I^{reg}. At first transform equation (14) into

$$E_I^{reg} = -\frac{\pi^2}{4}\,\hbar c\, L^2\, \frac{d^3}{d\alpha^3} \sum_{l=1}^{\infty} \int_0^{\infty} \frac{dz}{1+z} \exp\left[-\frac{l}{d}\,\alpha\,\sqrt{1+z}\right]\ . \quad (15)$$

Afterwards perform the sum over l and then take the derivative with respect to α and bring this result to the form

$$E_I^{reg} = \frac{\pi^2 \hbar c L^2}{2d}\, \frac{d^2}{d\alpha^2}\, \frac{d/\alpha}{\exp[\alpha/d]-1}\ . \quad (16)$$

Hint:

$$\frac{y}{e^y - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} y^n\ , \quad (17)$$

where the B_n are the Bernoulli numbers and show

$$U(d,L) := \lim_{R\to\infty} \lim_{\alpha\to 0} U^{reg}(d,L,R,\alpha) = -\frac{\pi^2}{720}\frac{\hbar c L^2}{d^3}, \quad (18)$$

which corresponds to equation (1).