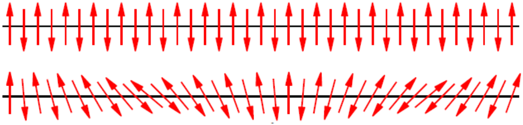
Quantum Field Theory for the Heisenberg-Antiferromagnet in 1D



Altland, A. and Simons, B., Condensed Matter Field Theory, Cambridge University Press, 2006

Heisenberg Hamiltonian

1D spin chain with periodic boundaries

$$H = J \sum_{j=1}^{N} \mathbf{S}_j \cdot \mathbf{S}_{j+1}$$

Ground state

- classical: spins oriented antiparallel (Néel State)
- qm: Néel State is not the ground state, not even an eigenstate!

$$H = J \sum_{j} \left(S_j^z S_{j+1}^z + \frac{1}{2} (S_j^- S_{j+1}^+ + S_j^+ S_{j+1}^-) \right)$$

- ground state of antiferromagnet shows quantum flactuations

The Spin-$\frac{1}{2}$ Chain

- $S_j^- S_{j+1}^+$ causes spinflip

$$S_j^- S_{j+1}^+ \left| \uparrow\downarrow \ldots \underset{j}{\uparrow}\downarrow \ldots \right\rangle = \left| \uparrow\downarrow \ldots \underset{j}{\downarrow}\uparrow \ldots \right\rangle$$

- exact solution via Bethe Ansatz
- groundstate: linear combination of states with half of the spins up
- energy spectrum is gapless

Outline

- Consider arbitrary s
- Path Integral of spin fields $\mathbf{n}(x, \tau)$
- Field Theory because $\xi >> a$ at low temperature

  ξ :correlation length, a: lattice constant

  $\rightsquigarrow$ just long length scales are interesting
- Map on non linear sigma model + topological term
- Difference between integer and half integer s

Repetition

- Partition Function:

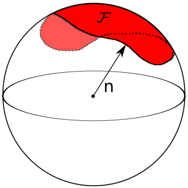
$$Z = \int \mathcal{D}[\mathbf{n}]\delta(\mathbf{n}^2 - 1)\mathrm{e}^{-S[\mathbf{n}]}$$

- Coherent states:

$$Spin : SU(2)/U(1) \simeq S^2$$

$$|\mathbf{n}\rangle = \mathrm{e}^{-i\theta\mathbf{m}\cdot S}|s, -s\rangle$$

$$\langle\mathbf{n}|\,\mathbf{S}\,|\mathbf{n}\rangle = -s\mathbf{n}, \quad |\mathbf{n}| = 1$$



- Euclidean Action:

$$S[\mathbf{n}] = \underbrace{\int_0^\beta d\tau \left(\langle\mathbf{n}| \frac{d}{d\tau} |\mathbf{n}\rangle\right.}_{\text{Berry Phase } = -is\mathcal{F}} \left. + \langle\mathbf{n}|\,H\,|\mathbf{n}\rangle \right)$$

- Heisenberg Hamiltonian:

$$\langle\mathbf{n}|\,H\,|\mathbf{n}\rangle = J\sum_j \langle\mathbf{n}|\,\mathbf{S}_j \cdot \mathbf{S}_{j+1}\,|\mathbf{n}\rangle = Js^2 \sum_j \mathbf{n}(j)\cdot\mathbf{n}(j+1)$$

Action of QM Heisenberg Antiferromagnet

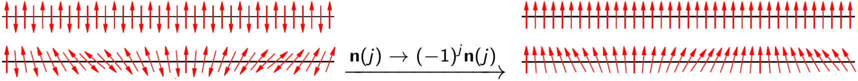
$$S[\mathbf{n}] = -is \sum_{j=1}^{N} \int_0^\beta d\tau \; \mathbf{A}(\mathbf{n}(j)) \cdot \partial_\tau \mathbf{n}(j) + \int_0^\beta d\tau \; Js^2 \sum_{j=1}^{N} \mathbf{n}(j) \cdot \mathbf{n}(j+1)$$

$$\text{with} \quad \nabla \times \mathbf{A} = \mathbf{n} \quad \Rightarrow \quad \int_\Gamma \mathbf{A} \cdot d\mathbf{n} = \int_F \left(\nabla \times \mathbf{A}\right) \cdot \mathbf{n} \; df = \mathcal{F}$$

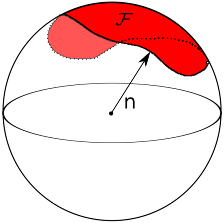
- Field Theory for $\mathbf{n}(x, \tau)$ with $x = aj$

  $\xi >> a \rightsquigarrow a \to 0$

- assume short range Néel order $\rightsquigarrow$ staggering of the spins



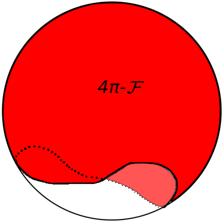
Berry Phase



$$-is\sum_{j=1}^{N}\int_0^\beta d\tau\;\mathbf{A}(\mathbf{n}(j))\cdot\partial_\tau\mathbf{n}(j)$$

$$\rightarrow\quad -is\sum_{j=1}^{N}\int_0^\beta d\tau\;\mathbf{A}((-1)^j\mathbf{n}(j))\cdot\partial_\tau(-1)^j\mathbf{n}(j)$$

$$=\quad -is\sum_{j=1}^{N}(-1)^j\int_0^\beta d\tau\;\mathbf{A}(\mathbf{n}(j))\cdot\partial_\tau\mathbf{n}(j)$$



$$\sum_{j=1}^{N}(-1)^j\mathbf{A}(\mathbf{n}(j))\cdot\partial_\tau\mathbf{n}(j)=\sum_{j=1}^{N/2}\Big[\mathbf{A}(\mathbf{n}(2j))\cdot\partial_\tau\mathbf{n}(2j)-\mathbf{A}(\mathbf{n}(2j-1))\cdot\partial_\tau\mathbf{n}(2j-1)\Big]$$

$$\mathbf{A}(\mathbf{n}(2j)) \cdot \partial_\tau \mathbf{n}(2j) - \mathbf{A}(\mathbf{n}(2j-1)) \cdot \partial_\tau \mathbf{n}(2j-1)$$

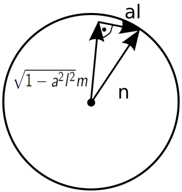
$$\overset{\mathbf{n}(2j)-\mathbf{n}(2j-1)=\delta\mathbf{n}(2j) \quad \leadsto \quad \text{small variation}}{}$$

$$
\begin{aligned}
&= \delta\Big(\mathbf{A}(\mathbf{n}(2j)) \cdot \partial_\tau \mathbf{n}(2j)\Big) \\
&= \delta\big(A_\alpha \partial_\tau n_\alpha\big) \\
&= (\partial_\beta A_\alpha)\delta n_\beta \partial_\tau n_\alpha + A_\alpha \partial_\tau \delta n_\alpha \\
&= (\partial_\alpha A_\beta)\partial_\tau n_\beta \delta n_\alpha - \partial_\tau A_\alpha \delta n_\alpha \\
&= \Big(\partial_\alpha A_\beta - \partial_\beta A_\alpha\Big)\partial_\tau n_\beta \delta n_\alpha \\
&= \Big(\big(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}\big)\partial_\mu A_\nu\Big)\partial_\tau n_\beta \delta n_\alpha \\
&= \Big(\varepsilon_{\gamma\alpha\beta}\varepsilon_{\gamma\mu\nu}\partial_\mu A_\nu\Big)\partial_\tau n_\beta \delta n_\alpha \qquad \varepsilon_{\gamma\mu\nu}\partial_\mu A_\nu = (\nabla \times \mathbf{A})_\gamma = n_\gamma \\
&= \varepsilon_{\gamma\alpha\beta} n_\gamma \partial_\tau n_\beta \delta n_\alpha \\
&= -\big(\mathbf{n} \times \partial_\tau \mathbf{n}\big) \cdot \delta \mathbf{n}
\end{aligned}
$$

Action of the staggered spin field $\mathbf{n}$

$$S[\mathbf{n}] = is \sum_{j=1}^{N/2} \int_0^\beta d\tau\, \delta\mathbf{n}(2j) \cdot \big(\mathbf{n}(2j) \times \partial_\tau \mathbf{n}(2j)\big) - Js^2 \int_0^\beta d\tau\, \sum_{j=1}^{N} \mathbf{n}(j) \cdot \mathbf{n}(j+1)$$

- divide $\mathbf{n}$ into slow and fast fluctuating part
- $\mathbf{n}(j) = \sqrt{1 - a^2 \mathbf{l}_j^2}\, \mathbf{m}_j + (-1)^j a\, \mathbf{l}_j$
- constraint: $\mathbf{m}_j \cdot \mathbf{l}_j = 0$ and $\mathbf{m}_j^2 = 1$



- expansions in orders of a:
  - $\sqrt{1 - a^2 \mathbf{l}_j^2} = 1 - \frac{1}{2} a^2 \mathbf{l}_j^2 + \mathcal{O}(a^4)$
  - $\mathbf{m}_{j+1} = \mathbf{m}_j + a\partial_x \mathbf{m}_j + \frac{a^2}{2} \partial_x^2 \mathbf{m}_j + \mathcal{O}(a^3)$
  - $a\mathbf{l}_{j+1} = a\mathbf{l}_j + a^2 \partial_x \mathbf{l}_j + \mathcal{O}(a^3)$

Berry Phase:

$$\delta\mathbf{n}(2j)\cdot\big(\mathbf{n}(2j)\times\partial_\tau\mathbf{n}(2j)\big)=a(\partial_x\mathbf{m}_{2j}+2\mathbf{l}_{2j})\cdot(\mathbf{m}_{2j}\times\partial_\tau\mathbf{m}_{2j})$$

$$\begin{aligned}\delta\mathbf{n}(2j) &= \mathbf{n}(2j)-\mathbf{n}(2j-1)\\ &= \sqrt{1-a^2\mathbf{l}_{2j}^2}\,\mathbf{m}_{2j}-\sqrt{1-a^2\mathbf{l}_{2j-1}^2}\,\mathbf{m}_{2j-1}+a(\mathbf{l}_{2j}+\mathbf{l}_{2j-1})\\ &= \mathbf{m}_{2j}-\mathbf{m}_{2j}+a\partial_x\mathbf{m}_{2j}+2a\mathbf{l}_{2j}+\mathcal{O}(a^2)\\ &= a(\partial_x\mathbf{m}_{2j}+2\mathbf{l}_{2j})+\mathcal{O}(a^2)\end{aligned}$$

$$\begin{aligned}\mathbf{n}(2j) &= \sqrt{1-a^2\mathbf{l}_{2j}^2}\,\mathbf{m}_{2j}+a\mathbf{l}_{2j}=\mathbf{m}_{2j}+\mathcal{O}(a)\\ \Rightarrow\ \mathbf{n}(2j)\times\partial_\tau\mathbf{n}(2j) &= \mathbf{m}_{2j}\times\partial_\tau\mathbf{m}_{2j}+\mathcal{O}(a)\end{aligned}$$

Interaction term:

$$\begin{aligned}
\mathbf{n}(j) &= \sqrt{1-a^2\mathbf{l}_j^2}\,\mathbf{m}_j + (-1)^j a\mathbf{l}_j \approx \left(1-\frac{a^2}{2}\mathbf{l}_j^2\right)\mathbf{m}_j + (-1)^j a\mathbf{l}_j \\
\mathbf{n}(j+1) &\approx \left(1-\frac{a^2}{2}\mathbf{l}_j^2\right)\mathbf{m}_j + a\partial_x\mathbf{m}_j + \frac{a^2}{2}\partial_x^2\mathbf{m}_j + a(-1)^{j+1}(\mathbf{l}_j + a\partial_x\mathbf{l}_j)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{n}(j)\cdot\mathbf{n}(j+1) \\
\approx\ & \left(1-\frac{a^2}{2}\mathbf{l}_j^2\right)^2\mathbf{m}_j^2 + a^2(-1)^j\mathbf{l}_j\cdot\partial_x\mathbf{m}_j + \frac{a^2}{2}\mathbf{m}_j\cdot\partial_x^2\mathbf{m}_j - a^2\mathbf{l}_j^2 - (-1)^j a^2\mathbf{m}_j\cdot\partial_x\mathbf{l}_j \\
\approx\ & 1 - a^2\mathbf{l}_j^2 + \frac{a^2}{2}\mathbf{m}_j\cdot\partial_x^2\mathbf{m}_j + a^2(-1)^j(\mathbf{l}_j\cdot\partial_x\mathbf{m}_j - \mathbf{m}_j\cdot\partial_x\mathbf{l}_j) - a^2\mathbf{l}_j^2 \\
\approx\ & \frac{a^2}{2}\mathbf{m}_j\cdot\partial_x^2\mathbf{m}_j - 2a^2\mathbf{l}_j^2
\end{aligned}$$

$$\partial_x(\mathbf{m}_j\cdot\mathbf{l}_j)=0 \quad\Rightarrow\quad \mathbf{l}_j\cdot\partial_x\mathbf{m}_j=-\mathbf{m}_j\cdot\partial_x\mathbf{l}_j$$

$$\sum_{j=1}^{N}(-1)^j\mathbf{l}_j\cdot\partial_x\mathbf{m}_j = \sum_{j=1}^{\frac{N}{2}}(\mathbf{l}_{2j}\cdot\partial_x\mathbf{m}_{2j}-\mathbf{l}_{2j-1}\cdot\partial_x\mathbf{m}_{2j-1}) = a\sum_{j=1}^{\frac{N}{2}}\partial_x(\mathbf{l}_{2j-1}\cdot\partial_x\mathbf{m}_{2j-1})$$

Limit $a \to 0$

Action $S[\mathbf{m}, \mathbf{l}]$:

$$is \sum_{j=1}^{N/2} a \int_0^\beta d\tau \, (\partial_x \mathbf{m}_{2j} + 2\mathbf{l}_{2j}) \cdot (\mathbf{m}_{2j} \times \partial_\tau \mathbf{m}_{2j}) - Js^2 \int_0^\beta d\tau \, \sum_{j=1}^{N} \left[\frac{a^2}{2} \mathbf{m}_j \cdot \partial_x^2 \mathbf{m}_j - 2a^2 \mathbf{l}_j^2 \right]$$

$$a \to 0: \quad \sum_{j=1}^{N} a \to \int dx \leftarrow \sum_{j=1}^{N/2} 2a$$

$$S[\mathbf{m}, \mathbf{l}] = is \int_0^\beta d\tau \int dx \left(\frac{1}{2} \partial_x \mathbf{m} + \mathbf{l} \right) \cdot (\mathbf{m} \times \partial_\tau \mathbf{m}) - \frac{aJs^2}{2} \int_0^\beta d\tau \int dx \left[\mathbf{m} \cdot \partial_x^2 \mathbf{m} - 4\mathbf{l}^2 \right]$$

$$= \int_0^\beta d\tau \int dx \left[i\frac{s}{2} \mathbf{m} \cdot (\partial_\tau \mathbf{m} \times \partial_x \mathbf{m}) + \frac{aJs^2}{2} (\partial_x \mathbf{m})^2 + is\mathbf{l} \cdot (\mathbf{m} \times \partial_\tau \mathbf{m}) + 2aJs^2 \mathbf{l}^2 \right]$$

$$\mathbf{m} = \mathbf{m}(x, \tau) \text{ and } \mathbf{l} = \mathbf{l}(x, \tau)$$

Integration over $\mathbf{l}$

Integrate out the fast flactuating part $\mathbf{l}$ in partition function

$$
\begin{aligned}
Z &= \int \mathcal{D}[\mathbf{m}]\mathcal{D}[\mathbf{l}]\delta(\mathbf{m}^2-1)\delta(\mathbf{m}\cdot\mathbf{l})\mathrm{e}^{-S[\mathbf{m},\mathbf{l}]} \\
&= \int \mathcal{D}[\mathbf{m}]\delta(\mathbf{m}^2-1)\mathrm{e}^{-S_1[\mathbf{m}]}\int \mathcal{D}[\mathbf{l}]\delta(\mathbf{m}\cdot\mathbf{l})\mathrm{e}^{\int d\tau dx\left[-is\mathbf{l}\cdot(\mathbf{m}\times\partial_\tau\mathbf{m})-2aJs^2\mathbf{l}^2\right]} \\
&= \int \mathcal{D}[\mathbf{m}]\delta(\mathbf{m}^2-1)\mathrm{e}^{-S_1[\mathbf{m}]}\int \mathcal{D}[\mathbf{l}]\delta(\mathbf{m}\cdot\mathbf{l})\mathrm{e}^{\int d\tau dx\left[-\frac{1}{2}B\mathbf{l}^2+\mathbf{l}\cdot\mathbf{b}\right]}
\end{aligned}
$$

$$
B := 4aJs^2 \quad \mathbf{b} := -is(\mathbf{m}\times\partial_\tau\mathbf{m})
$$

Integration over $\mathbf{l}$

$$B := 4aJs^2 \qquad \mathbf{b} := -is(\mathbf{m}\times\partial_\tau \mathbf{m})$$

$$\begin{aligned}
&\int \mathcal{D}[\mathbf{l}]\delta(\mathbf{m}\cdot\mathbf{l})\mathrm{e}^{\int d\tau dx\left[-\frac{1}{2}B\mathbf{l}^2+\mathbf{l}\cdot\mathbf{b}\right]} \\
=\ & \int\left(\prod_j\prod_k d^3 l_{j,k}\right)\delta(\mathbf{m}_{j,k}\cdot\mathbf{l}_{j,k})\,\mathrm{e}^{\sum_j a\sum_k \delta\tau\left[-\frac{1}{2}B\mathbf{l}^2+\mathbf{l}_{j,k}\cdot\mathbf{b}_{j,k}\right]} \\
=\ & \prod_j\prod_k \int d^2 l_{j,k}\ \mathrm{e}^{-\frac{1}{2}a\delta\tau B\mathbf{l}^2+a\delta\tau\mathbf{l}_{j,k}\cdot\mathbf{b}_{j,k}} \\
=\ & \prod_j\prod_k \frac{2\pi}{a\delta\tau B}\ \mathrm{e}^{\frac{1}{2}a\delta\tau\frac{1}{B}\mathbf{b}_{j,k}^2} \\
=\ & \mathcal{C}\ \mathrm{e}^{-\int d\tau dx\frac{(\partial_\tau\mathbf{m})^2}{8aJ}}
\end{aligned}$$

in the last step use: $(\mathbf{m}\times\partial_\tau\mathbf{m})^2 = (\partial_\tau\mathbf{m})^2$

Non Linear Sigma Model

$$
\begin{aligned}
S[\mathbf{m}] &= \int_0^\beta d\tau \int dx \left[\frac{aJs^2}{2} (\partial_x \mathbf{m})^2 + \frac{1}{8aJ} (\partial_\tau \mathbf{m})^2 + i\frac{s}{2} \mathbf{m} \cdot (\partial_\tau \mathbf{m} \times \partial_x \mathbf{m}) \right] \\
&= \int d\tau dx \, \frac{1}{2g} \left(v_s (\partial_x \mathbf{m})^2 + \frac{1}{v_s} (\partial_\tau \mathbf{m})^2 \right) + i\frac{s}{2} \int d\tau dx \, \mathbf{m} \cdot (\partial_\tau \mathbf{m} \times \partial_x \mathbf{m})
\end{aligned}
$$

where $g = \frac{2}{s}$ and $v_s = 2aJs$

Rescaling: $v_s \, \tau = y$, $\frac{1}{v_s} \partial_\tau = \partial_y$

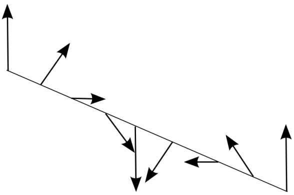
$$
\begin{aligned}
S[\mathbf{m}] &= \int dx^2 \, \frac{1}{2g} \left((\partial_x \mathbf{m})^2 + (\partial_y \mathbf{m})^2 \right) + i\frac{s}{2} \int dx^2 \, \mathbf{m} \cdot (\partial_y \mathbf{m} \times \partial_x \mathbf{m}) \\
&= \text{Non linear Sigma model} + \text{Topological term}
\end{aligned}
$$

Topological term

$$\begin{aligned} & i\frac{s}{2}\int dx^2\ \mathbf{m}\cdot(\partial_y\mathbf{m}\times\partial_x\mathbf{m}) \\ = \quad & i2\pi s\,\frac{1}{8\pi}\int dx^2\ \varepsilon_{ij}\mathbf{m}\cdot(\partial_i\mathbf{m}\times\partial_j\mathbf{m}) \\ = \quad & i2\pi s\,\mathcal{Q} \end{aligned}$$

- $\mathcal{Q}\in\mathbb{Z}$
- Winding number of spin field

$$\mathrm{e}^{-i2\pi s\,\mathcal{Q}} = \begin{cases} 1 & \text{s integer} \\ (-1)^{\mathcal{Q}} & \text{s half integer} \end{cases}$$



Haldane's conjecture

- Integer spin
  - NLσM without topological term
  - finite correlation length
  - Gap in energy spectrum
- Half integer spin
  - NLσM with topological term
  - Gapless energy spectrum
  - Bethe Ansatz proves this for $s = \frac{1}{2}$

References

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