Topological Excitations II:
Hopf terms and Anyons

ITP-III Seminar: Quantum Field Theory of low-dimensional Systems

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July 17, 2012

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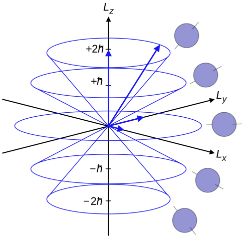
Overview

Overview: Spin and statistics in $d+1$ dimensions

Spin in $3+1$ dimensions [Kha05, Ler92]

- In $3+1$ ($d+1$, $d \geq 3$) dimensions the spin angular momentum algebra is non-commutative:

$$[S_i, S_j] = i\varepsilon_{ijk} S_k \quad \text{where} \quad i,j,k = 1,2,3$$



↪ [Wik12a]

- Since $\left[\mathbf{S}^2, S_3\right] = 0$ we find for the spin eigenstates

$$\mathbf{S}^2 \left|s,m\right\rangle = S(S+1)\left|S,m\right\rangle \quad \text{and} \quad S_3\left|S,m\right\rangle = m\left|S,m\right\rangle$$

- By means of raising and lowering operators $S_\pm$ and demanding a positive norm one finds

$$2S \in \mathbb{N}_0 \quad \Rightarrow \quad S = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

Spin in $2+1$ dimensions: "Any" difference? [Kha05, Ler92]

- There is only one rotational axis (perpendicular to the plane)
  $\hookrightarrow$ Only one component spin operator $\mathbf{S} \equiv S_3$
- Obviously there are no commutation relations
- Hence there is no restriction regarding the spin quantum number S:

$$\boxed{S \in \mathbb{R}}$$

Motivation: Spin-statistics theorem

In $3+1$ dimensions:

$$\text{Integer spin particles} \quad \Rightarrow \quad \text{Bosonic statistics}$$
$$\text{Half-integer spin particles} \quad \Rightarrow \quad \text{Fermionic statistics}$$

In $2+1$ dimensions:

$$\text{"Any" spin particles} \quad \Rightarrow \quad \text{"Any" statistics ("Anyons") ?}$$

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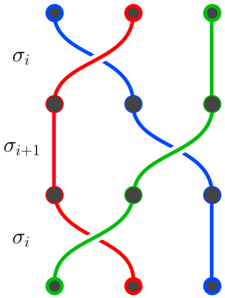
The braid group $\mathcal{B}_N$ [Kha05]

The braiding of strands motivates the definition of the braid group $\mathcal{B}_N$:

Informal definition: Braid group $\mathcal{B}_N$

- N strands attached to N start- and endpoints
- They may be braided arbitrarily
- Group operation $=$ Concatenation of braids
- Isotopic strands are equivalent
- Generators σ_i $=$ Crossing of adjacent strands



- Example: $\mathcal{B}_1 = \{\mathbb{1}\}$, $\mathcal{B}_2 \cong \mathbb{Z}$
- $\mathcal{B}_N$ ($N \geq 3$) is finitely generated, non-abelian and of infinite order
- $\mathcal{B}_N$ is a generalization of $\mathcal{S}_N$ insofar as $\mathcal{S}_N \cong \mathcal{B}_N \big/ (\sigma_i^2 = 1)$, i.e. there is a surjective group homomorphism $\mathcal{B}_N \to \mathcal{S}_N$ ("forget the strands")

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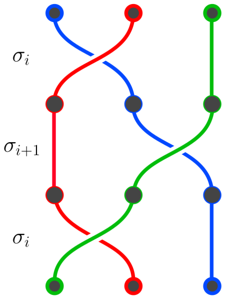

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Homotopy [Nak03]

- Two continuous functions are homotopic if there is a continuous deformation from one into the other
- Most intuitive example: Deformation of a path $\Gamma : [0,1] \longrightarrow Y$ with fixed endpoints →

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Definition: Homotopy

Let X and Y be topological spaces and $f, g : X \to Y$ two continuous mappings. A continuous function

$$H : X \times [0,1] \longrightarrow Y$$

is called homotopy if $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. In this case f and g are said to be homotopic, write $f \simeq g$.

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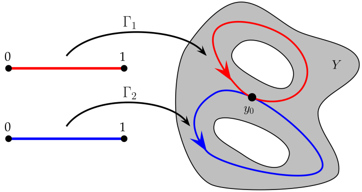
Homotopy groups: Motivation [Nak03]

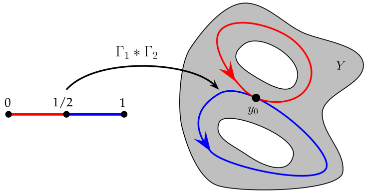
Consider …

- an oriented, compact 2-manifold Y
- with basepoint $y_0 \in Y$
- and two loops Γ_1, Γ_2 with common start- and endpoint $\Gamma_1(0) = \Gamma_1(1) = y_0 = \Gamma_2(0) = \Gamma_2(1)$.



The product $\Gamma_1 * \Gamma_2$ of two loops is their composition:

$$(\Gamma_1 * \Gamma_2)(t) := \begin{cases} \Gamma_1(2t) & t \in [0, 1/2] \\ \Gamma_2(2t-1) & t \in (1/2, 1] \end{cases}$$



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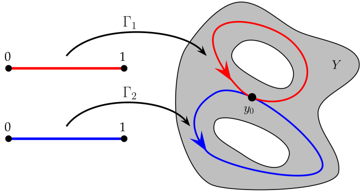

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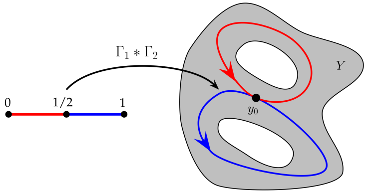

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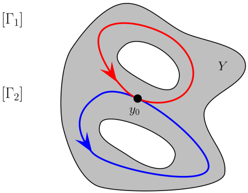
Define the set of all continuous loops

$$F = \{\Gamma : [0,1] \to Y \mid \Gamma(0) = y_0 = \Gamma(1)\}$$

Write $\Gamma_1 \simeq \Gamma_2$ for homotopic loops via H and $H(0,t) = y_0 = H(1,t)$ for all $t \in [0,1]$.



A group operation $\bullet$ on $F/_{\simeq}$ is defined as

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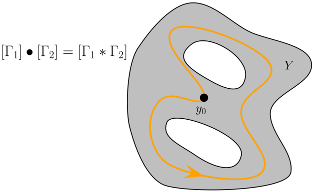

This yields the first homotopy group or fundamental group

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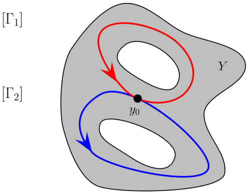

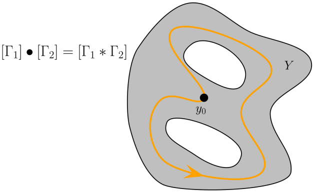

Homotopy groups: Definition [Nak03]

A slightly more general approach leads to the *p*-th homotopy group:

Definition: Homotopy groups

Let Y be a topological space with basepoint y_0 and S^p the p-sphere with basepoint x_0. The set

$$\pi_p(Y, y_0) := \left(\{\Gamma : S^p \to Y \,|\, \Gamma(x_0) = y_0\} /_{\simeq}, \bullet\right)$$

of all homotopy classes of maps that map x_0 to y_0 equipped with the composition $\bullet$ of maps[1] is called *p*-th homotopy group of Y.

Intuitive examples for homotopy groups:

$$\pi_1\left(S^1\right) \cong \mathbb{Z} \qquad \pi_1\left(S^2\right) \cong \{\mathbb{1}\} \qquad \pi_2\left(S^2\right) \cong \mathbb{Z}$$

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Configuration space: Construction [Kha05, Ler92]

Configuration space of ...

- $N = 1$ particle in d dimensions: $\mathcal{M}_1^d = \mathbb{R}^d$
- $N > 1$ indistinguishable particles in d dimensions: $\mathcal{M}_N^d = \left(\mathbb{R}^d\right)^N$?

This is not correct! On a physical level permutations are identified:

$$(\mathbf{x}_1, \dots, \mathbf{x}_N) \sim \left(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}\right) \quad \text{where} \quad \sigma \in \mathcal{S}_N$$

We assume hard-core particles (!), i.e. they cannot occupy the same position. Therefore the diagonal

$$\Delta := \{(\mathbf{x}_1, \dots, \mathbf{x}_2) \,|\, \mathbf{x}_i = \mathbf{x}_j \text{ for some } i \neq j\}$$

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$$\mathcal{M}_N^d := \left[\left(\mathbb{R}^d\right)^N \setminus \Delta\right] / \mathcal{S}_N$$

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$\mathcal{M}_N^d$ is not simply connected! Indeed, one can show:

Proposition: Fundamental group of $\mathcal{M}_N^d$

The fundamental group π_1 of the configuration space $\mathcal{M}_N^d$ in $d+1$ dimensional spacetime (d space dimensions) is

$$\pi_1\left(\mathcal{M}_N^d\right) \cong \begin{cases} \mathcal{S}_N & \text{for} \quad d \geq 3 \\ \mathcal{B}_N & \text{for} \quad d = 2 \end{cases}$$

We immediately realize:

- $d \geq 3$: there are 2 1d-representations of $\mathcal{S}_N \hookrightarrow$ Fermions & Bosons
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The path integral: Reminder [Kha05, Ler92]

Consider a system described by $\mathcal{M}_N^d$ and configurations $q_1, q_2 \in \mathcal{M}_N^d$.
The transition amplitude K is obtained via the path integral:

Reminder: Feynman path integral

$$K(q_2, t_2; q_1, t_1) = \chi \int\limits_{q(t_i)=q_i} \mathcal{D}[q] \exp\left\{ \frac{i}{\hbar} \int_{t_1}^{t_2} \mathrm{d}\tau \mathcal{L}\left[q(\tau), \dot{q}(\tau)\right] \right\}$$

Where χ is an arbitrary global phase:
independent of homotopic (!) paths $\hookrightarrow$ unobservable

We note: There is no rule (regarding the quantization) demanding χ to be the same for non-homotopic paths:

$$\chi \longrightarrow \chi(\alpha) \qquad \text{where} \qquad \alpha \in \pi_1\left(\mathcal{M}_N^d\right)$$

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where $q_\alpha \in \alpha$.

Let $\alpha_i = [\gamma_i] \in \pi_1\left(\mathcal{M}_N^d\right)$. Consider a propagation along ...

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The path integral: Statistics

The representation χ determines the statistics. We find:

$d \geq 3$ **dimensions**	$d = 2$ **dimensions**
$\pi_1\left(\mathcal{M}_N^d\right) \cong \mathcal{S}_N$	$\pi_1\left(\mathcal{M}_N^d\right) \cong \mathcal{B}_N$
There are two 1d-irreps $\chi_1 : \sigma \mapsto 1$, $\chi_2 : \sigma \mapsto (-1)^\sigma$ for all $\sigma \in \mathcal{S}_N$.	There are ∞ many 1d-irreps $\chi_\theta : \sigma_j \mapsto e^{i\theta}$ for all $1 \leq j \leq N-1$.
There are two types of statistics: $\chi_1 \leftrightarrow$ Bosons, $\chi_2 \leftrightarrow$ Fermions	There is any statistics: $\chi_{\theta=0} \leftrightarrow$ Bosons, $\chi_{\theta\neq 0,\pi} \leftrightarrow$ Anyons, $\chi_{\theta=\pi} \leftrightarrow$ Fermions

Overview: The $O(3)$ nonlinear σ-Model revisited

The NLσM: Definition [Raj87, Pol75, BP75]

The NLσM emerges as a continuum theory for the 2d-Heisenberg model:

Definition: NLσM in 2+1 dimensions

Consider three real scalar fields $\Phi_i : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} : (t, \vec{x}) \mapsto \Phi_i(t, \vec{x}),\ i = 1, 2, 3$, with the constraint $\sum_i \Phi_i^2(t, \vec{x}) \equiv \mathbf{\Phi} \cdot \mathbf{\Phi} = 1$ for all $(t, \vec{x})$.
The lagrangian density

$$\mathcal{L} = \frac{1}{2} \sum_{\mu=0}^{2} \sum_{i=1}^{3} (\partial_\mu \Phi_i) \cdot (\partial^\mu \Phi_i) \equiv \frac{1}{2} (\partial_\mu \mathbf{\Phi}) \cdot (\partial^\mu \mathbf{\Phi})$$

defines a field theory in 2+1 dimensional spacetime, namely the NLσM.

The NLσM: Action and energy [Raj87, Pol75, BP75]

From the Lagrangian density we find the action

$$S[\mathbf{\Phi}] = \frac{1}{2} \int \mathrm{d}^2 x \int \mathrm{d}t \, (\partial_\mu \mathbf{\Phi}) \cdot (\partial^\mu \mathbf{\Phi})$$

with the constraint $\mathbf{\Phi} \cdot \mathbf{\Phi} = 1 \hookrightarrow$ Symmetry group: $O(3)$

A Legendre transform yields the energy functional

$$E = \frac{1}{2} \int \mathrm{d}^2 x (\partial_k \mathbf{\Phi})^2 \equiv \frac{1}{2} \sum_{k=1}^{2} \sum_{i=1}^{3} \int \mathrm{d}^2 x (\partial_k \Phi_i)^2 \quad \geq \quad 0$$

Ground state: $E = 0 \Leftrightarrow \mathbf{\Phi}(\vec{x}) \equiv \mathbf{\Phi}_0 = \text{const.}$, where $|\mathbf{\Phi}_0| = 1$

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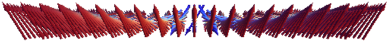
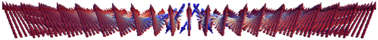
Solitons with $|Q| = 1$: Skyrmions [Raj87, WZ83]

Skyrmion ($Q = 1$)



$$\mathbf{\Phi}_S(r,\varphi) = \frac{1}{1+\left(\frac{r}{2\lambda}\right)^2}\begin{bmatrix} \frac{r}{\lambda}\cos\varphi \\ \frac{r}{\lambda}\sin\varphi \\ \left(\frac{r}{2\lambda}\right)^2 - 1 \end{bmatrix}$$

Antiskyrmion ($Q = -1$)



$$\mathbf{\Phi}_{\bar{S}}(r,\varphi) = \frac{1}{1+\left(\frac{\lambda}{2r}\right)^2}\begin{bmatrix} \frac{\lambda}{r}\cos\varphi \\ -\frac{\lambda}{r}\sin\varphi \\ \left(\frac{\lambda}{2r}\right)^2 - 1 \end{bmatrix}$$

One step further: Dynamics

- So far static solutions were considered:
  - $\mathbf{\Phi} = \mathbf{\Phi}(\vec{x})$, i.e. $\mathbf{\Phi} : S^2_{(phy)} \longrightarrow S^2_{(int)}$
  - $\pi_2(S^2) \cong \mathbb{Z} \hookrightarrow$ Pontryagin number Q
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	π_1	π_2	π_3	π_4	π_5	π_6
S^0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}
S^3	0	0	Z	Z_2	Z_2	Z_{12}
S^4	0	0	0	Z	Z_2	Z_2
S^5	0	0	0	0		

$\hookrightarrow$ [Wik12c]

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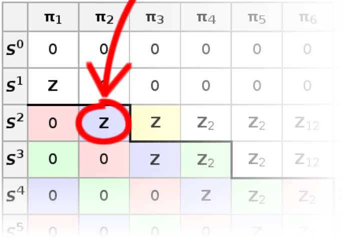
Classifying static solutions (Pontryagin number)



	π_1	π_2	π_3	π_4	π_5	π_6
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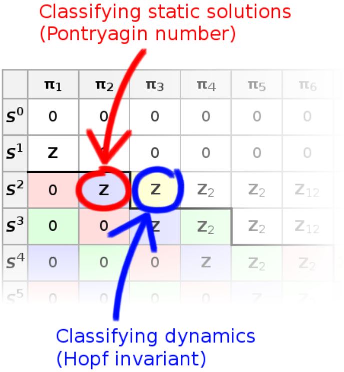

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Overview: Hopf term and anyonic excitations

Setting the scene [Fra98, WZ84, Wil70]

In the following, time-dependent solutions of the NLσM are considered:

$$\mathbf{\Phi} \,:\, \mathbb{R}_{\text{Time}} \times \mathbb{R}^2_{\text{Space}} \longrightarrow S^2_{(int)} \quad \text{where} \quad (t,\vec{x}) \mapsto \mathbf{\Phi}(t,\vec{x})$$

We demand $\lim\limits_{|\vec{x}|\to\infty} \mathbf{\Phi}(t,\vec{x}) = \mathbf{\Phi}_0$ and $\lim\limits_{|t|\to\infty} \mathbf{\Phi}(t,\vec{x}) = \mathbf{\Phi}_0$, therefore

$$\mathbf{\Phi} \,:\, \mathbb{R}\times\mathbb{R}^2 \cup \{\infty\} \cong S^3_{(phy)} \longrightarrow S^2_{(int)} \quad \text{where} \quad (t,\vec{x}) \mapsto \mathbf{\Phi}(t,\vec{x}),\ \infty \mapsto \mathbf{\Phi}_0$$

The configuration space for the NLσM is $\mathcal{M}_\sigma = \{\mathbf{\Phi} \,:\, S^2 \to S^2\}$.
Since $\delta Q[\mathbf{\Phi}] = 0$, $\mathcal{M}_\sigma$ decomposes into disjoint, connected components $\mathcal{M}^Q_\sigma$:

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The dynamics in the vacuum sector $\mathcal{M}^0_\sigma$ is described by[2]

$$\pi_1(\mathcal{M}^0_\sigma) \cong \pi_3(S^2) \cong \mathbb{Z}$$

[2] For details see e.g. [Fra98, WZ84] and especially [Wil70].

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Modifying the NLσM action [WZ83, Fra98, Hop31]

The skyrmion of the unmodified NLσM is a boson and has $S = 0$.

However, $\pi_1\left(\mathcal{M}_\sigma^0\right) \cong \pi_3(S^2) \cong \mathbb{Z}$ is non-trivial
$\hookrightarrow$ Add topological term to the action:

$$\tilde{S}[\mathbf{\Phi}] := S[\mathbf{\Phi}] + \theta H[\mathbf{\Phi}] \quad \text{where} \quad \theta \in \mathbb{R}$$

Note: Whether $\theta \neq 0$ in $\tilde{S}$ depends on the microscopic theory.

H is called Hopf invariant:

- $H[\mathbf{\Phi}] \in \mathbb{Z}$ for all $\mathbf{\Phi} \in \mathcal{M}_\sigma$
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The modified action and the path integral [WZ84]

Consider the propagator for initial and final state $\mathbf{\Phi}_0$ (Vacuum):

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K(\mathbf{\Phi}_0, -\infty; \mathbf{\Phi}_0, +\infty) &= \int\limits_{\text{Paths } \mathbf{\Phi}(t,\vec{x}) \text{ in } \mathcal{M}^0_\sigma} \mathcal{D}[\mathbf{\Phi}] \exp\left\{ iS[\mathbf{\Phi}] + i\theta H[\mathbf{\Phi}] \right\} \\
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The topological current and the gauge potential [WZ83]

The topological current

$$J^{\mu} := \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\varepsilon^{abc}\Phi_a\partial_\nu\Phi_b\partial_\lambda\Phi_c$$

is automatically conserved: $\partial_\mu J^\mu = 0$ ($\hookrightarrow$ not a dynamically conserved quantity)

Reminder: Topological charge = Pontryagin number

The topological charge is the well-known Pontryagin number

$$Q = \int \mathrm{d}^2x\, J^0 = \frac{1}{8\pi}\int \mathrm{d}^2x\, \varepsilon^{ij}\varepsilon^{abc}\Phi_a\partial_i\Phi_b\partial_j\Phi_c$$

Since $\partial_\mu J^\mu = 0$ there is a gauge potential A^μ such that

$$J^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$$

and the gauge freedom $A_\mu \to A_\mu - \partial_\mu\Lambda$.
Note that $A^\mu = A^\mu[\mathbf{\Phi}]$ depends nonlocally on $\mathbf{\Phi}$.

An analytic expression for the Hopf invariant [WZ83]

We may now give an analytic expression for the Hopf invariant:

Definition: Hopf invariant

Let $\mathbf{\Phi} : S^3_{(\mathrm{phy})} \longrightarrow S^2_{(\mathrm{int})}$. The Hopf invariant is defined by

$$H[\mathbf{\Phi}] := -\frac{1}{2\pi} \int_{\text{Spacetime}} \mathrm{d}^3 x \, J^\mu A_\mu \qquad \in \mathbb{Z}$$

From this expression we find easily

- H is gauge invariant
- H is homotopically invariant: $\delta H[\mathbf{\Phi}] = 0$

However, evaluating the above integral proves to be intricate!

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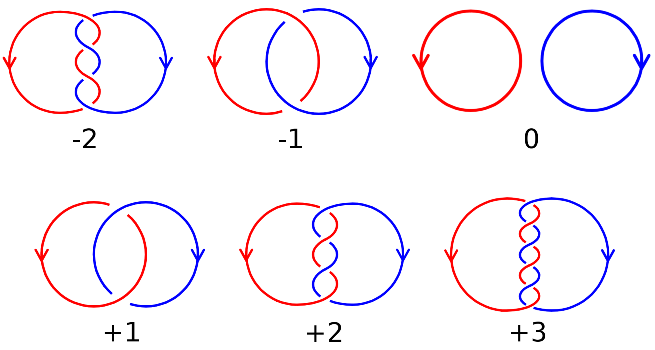
A side note: Linking numbers [Wik12d]



↪ [Wik12d]

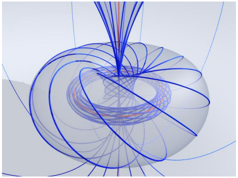
Proposition: Gauss integral representation

$$\mathrm{Link}\,[\gamma_1, \gamma_2] = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot (\mathrm{d}\mathbf{r}_1 \times \mathrm{d}\mathbf{r}_2)$$

Computing the Hopf invariant [Wil90, WZ83, Bae92]

Proposition: Preimage of points in S^2

Let $\mathbf{\Phi} : S^3 \longrightarrow S^2$ be a differentiable map. Then $\mathbf{\Phi}^{-1}(\mathbf{y}) \subset S^3$ is a collection of nonintersecting closed curves for almost every $\mathbf{y} \in S^2$.



$\hookrightarrow$ [Ci12]

"There is a deep theorem which equates the Hopf invariant to the linking number between two curves in $\mathbb{R}^3$" [WZ83]:

Proposition: Hopf invariant and linking numbers

Let $\mathbf{\Phi} : S^3 \longrightarrow S^2$ be a continuous mapping and $\mathbf{\Phi}_1, \mathbf{\Phi}_2 \in S^2$ two arbitrary fixed values. If γ_i denotes the closed curve given by $\mathbf{\Phi}^{-1}(\mathbf{\Phi}_i)$, it holds

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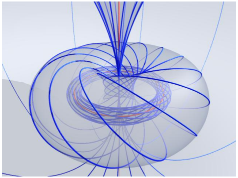

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Preparation: Creating Skyrmion-Antiskyrmion pairs

- Recall: $Q[\mathbf{\Phi}_1] = Q[\mathbf{\Phi}_2] \Leftrightarrow \mathbf{\Phi}_1 \simeq \mathbf{\Phi}_2$, $Q[\mathbf{\Phi}_1 * \mathbf{\Phi}_2] = Q[\mathbf{\Phi}_1] + Q[\mathbf{\Phi}_2]$
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Thus we find

$$Q[\mathbf{\Phi}_S * \mathbf{\Phi}_{\bar{S}}] = 1 + (-1) = 0 = Q[\mathbf{\Phi}_0] \quad \Rightarrow \quad \mathbf{\Phi}_S * \mathbf{\Phi}_{\bar{S}} \simeq \mathbf{\Phi}_0$$

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The spin of skyrmions [WZ83]

To derive the spin of a skyrmion we employ the following procedure:

Create $S\bar{S}$-pair $\Rightarrow$ Rotate S through 2π $\Rightarrow$ Annihilate $S\bar{S}$-pair

and track two fixed field values $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ in spacetime.

$\hookrightarrow$ The word lines γ_1 and γ_2 are linked once, thus

$$H[\mathbf{\Phi}] = \mathrm{Link}[\gamma_1, \gamma_2] = 1$$

We note: Fractional spin

The rotation of a spin-S particle results in a phase $e^{2\pi i S}$, therefore

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$\hookrightarrow$ The Hopf term endows the skyrmions with fractional spin!

Recall: S_z = Generator of rotations in 2D $\hookrightarrow$ $R(\omega) = e^{-i\omega \mathbf{S}/\hbar} \overset{\omega=-2\pi \mathbf{e}_z}{=} e^{2\pi i S_z/\hbar} = e^{2\pi i S}$

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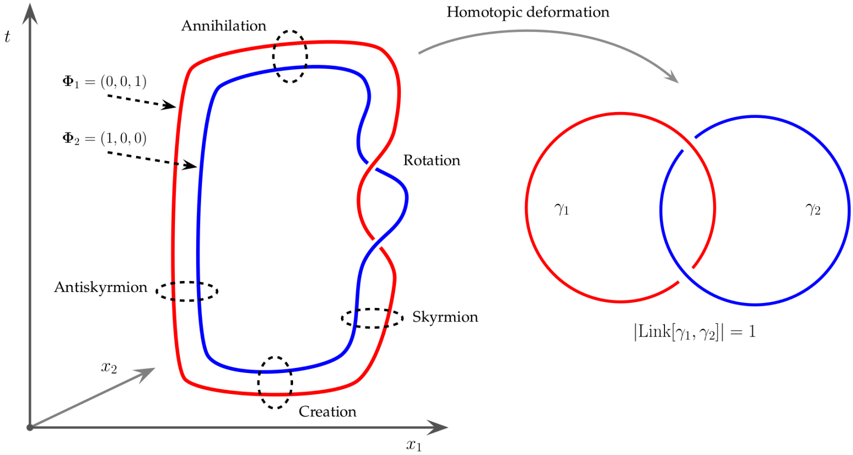

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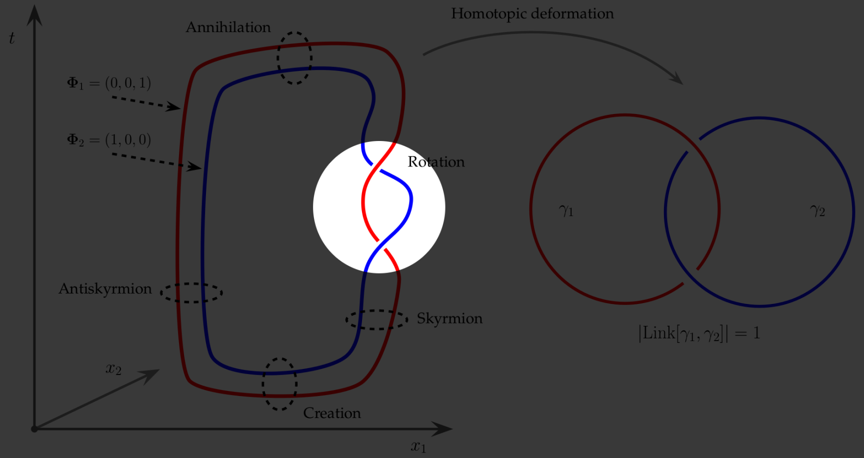

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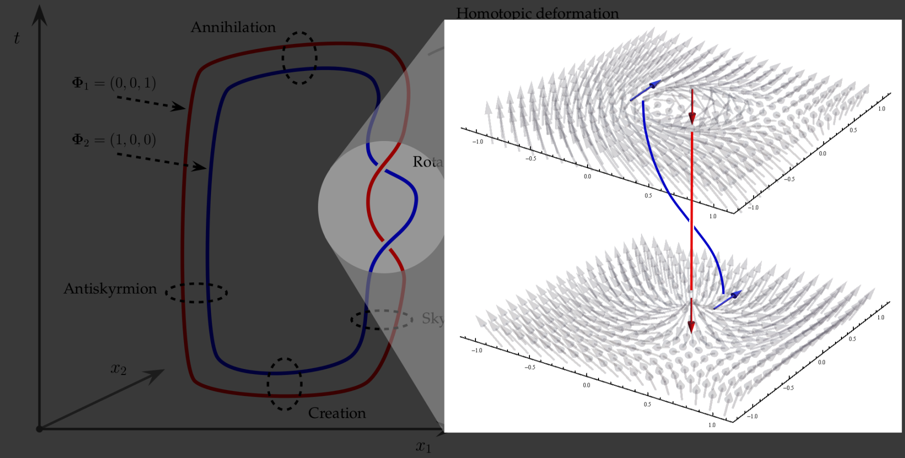

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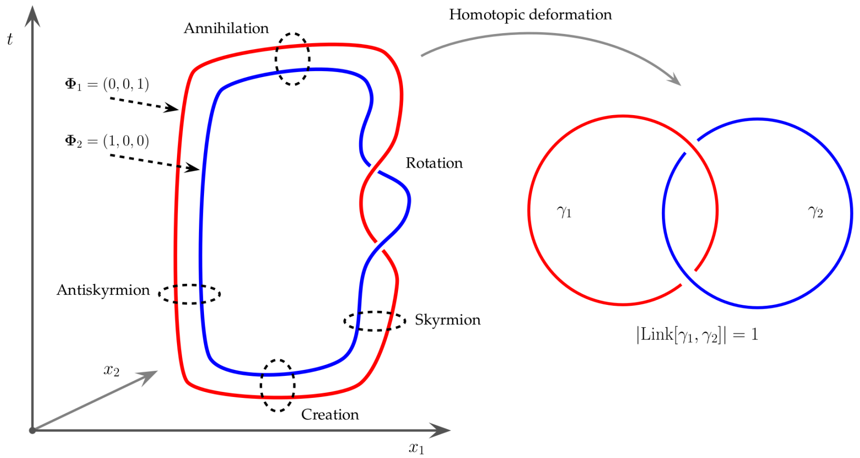

The spin of skyrmions [WZ83]

To derive the spin of a skyrmion we employ the following procedure:

Create $S\bar{S}$-pair $\Rightarrow$ Rotate S through 2π $\Rightarrow$ Annihilate $S\bar{S}$-pair

and track two fixed field values $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ in spacetime.

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We note: Fractional statistics

The interchange of two skyrmions results in a phase

$$e^{i\theta H[\Phi]} = e^{i\theta} \quad \text{where} \quad 0 \leq \theta \leq 2\pi$$

$\hookrightarrow$ The Hopf term endows the skyrmions with fractional statistics!

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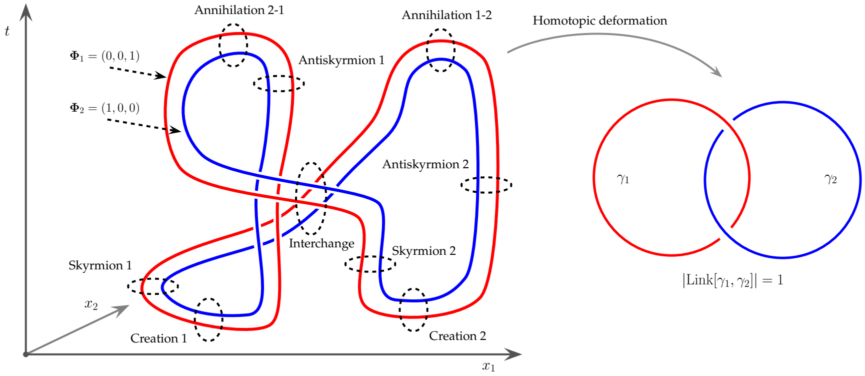

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Summary

What you should remember ...

Summary: Spin and statistics in $d+1$ dimensions

- First hint: Spin in $2+1$ dimensions not quantized $\hookrightarrow$ "Any" statistics?
- The configuration space for N hard-core particles is not simply connected:

$$\pi_1(\mathcal{M}_N^d) \cong \mathcal{S}_N \quad \text{for} \quad d \geq 3 \quad \text{and} \quad \pi_1(\mathcal{M}_N^d) \cong \mathcal{B}_N \quad \text{for} \quad d = 2$$

- The quantization by path integrals is not uniquely determined unless we choose a 1d-representation of $\pi_1(\mathcal{M}_N^d)$
- There are two 1d-representation for $\mathcal{S}_N \hookrightarrow$ Bosons, Fermions
- There are ∞ many 1d-representations for $\mathcal{B}_N \hookrightarrow$ Anyons

Summary: The $O(3)$ nonlinear σ-Model revisited

- We reviewed the NLσM: Action $S[\mathbf{\Phi}]$; Energy functional
- Finite energy $\hookrightarrow$ One-point compactification: $\mathbf{\Phi}\ :\ S^2_{(\mathrm{phy})} \longrightarrow S^2_{(\mathrm{int})}$
- Classification of static solutions by $\pi_2(S^2) \cong \mathbb{Z} \hookrightarrow$ Pontryagin number $Q[\mathbf{\Phi}]$
- Static solutions with $Q = \pm 1$ are called Skyrmions and Antiskyrmions
- In the unmodified NLσM they obey bosonic statistics and have $S = 0$

Summary: Hopf term and anyonic excitations

- Loop time evolution $\hookrightarrow$ One-point compactification: $\mathbf{\Phi} : S^3_{(\text{phy})} \longrightarrow S^2_{(\text{int})}$
- We added a topological term to the action: $\tilde{S}[\mathbf{\Phi}] := S[\mathbf{\Phi}] + \theta H[\mathbf{\Phi}]$
- The Hopf term $H : \pi_3(S^2) \longrightarrow \mathbb{Z}$ classifies the homotopy sectors
- $H[\mathbf{\Phi}]$ may be computed via ...
  - the topological current J^μ
  - the linking number $\text{Link}[\gamma_1, \gamma_2]$ of worldlines γ_i (easier!)
- Rotating a skyrmion yields $H[\mathbf{\Phi}] = 1 \hookrightarrow$ Fractional spin: $S = \frac{\theta}{2\pi} + k$
- Permuting two skyrmions yields $H[\mathbf{\Phi}] = 1 \hookrightarrow$ Fractional statistics: $e^{i\theta}$

$$\text{NL}\sigma\text{M} \quad + \quad \text{Hopf term} \quad \Rightarrow \quad \text{Skyrmions} \quad = \quad \text{Anyons}$$

That’s it!

Thank you for your attention.

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The symmetric group $\mathcal{S}_N$

Definition: Symmetric group $\mathcal{S}_N$

Let $X = \{1, 2, \ldots, N\}$. The set of all bijective functions $X \to X$ furnished with the composition as product is called symmetric group on X

$$\mathcal{S}_N := \mathrm{Aut}(X) = (\{f : X \to X \mid f \text{ bijective}\}, \circ)$$

- Example: $\sigma = (132) \to$
- $\mathcal{S}_N$ ($N \geq 3$) is finitely generated, non-abelian and of finite order
- There are two 1d-representations:

$$D_B : \mathcal{S}_N \ni \sigma \quad \mapsto \quad 1 \in \mathbb{C}$$
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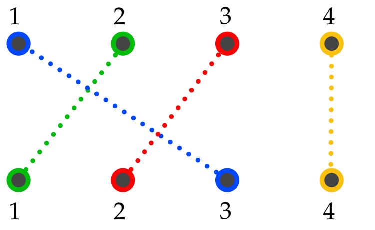
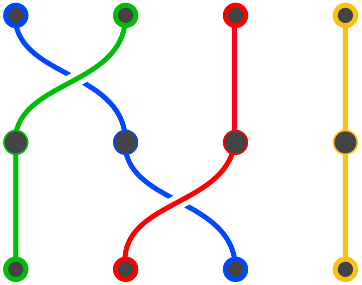
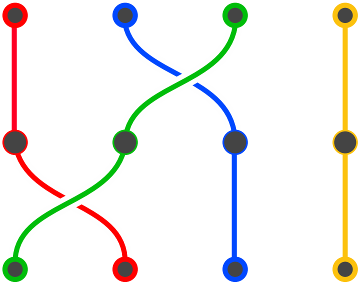


$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = (1\,3\,2)(4) \equiv (1\,3\,2)$$

The braid group $\mathcal{B}_N$ [Kha05]



σ_i σ_{i+1}



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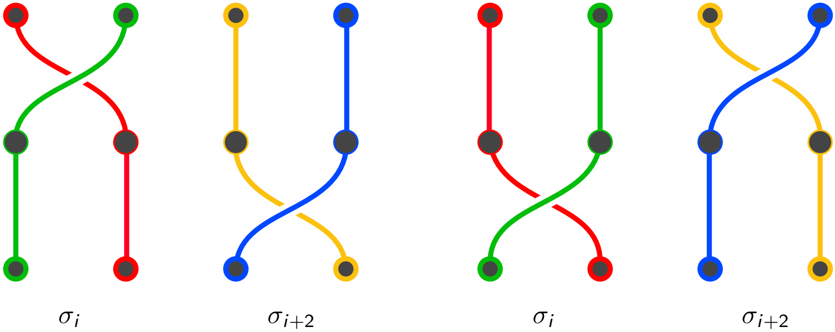

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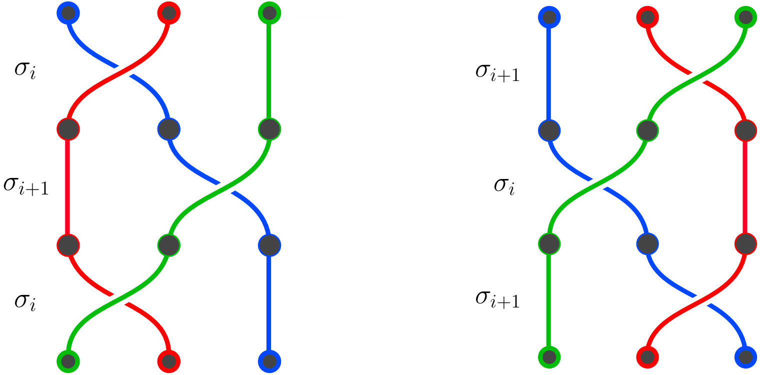

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The NLσM: Field equations and energy [Raj87, Pol75, BP75]

Introducing the lagrange multiplier $\lambda(t,\vec{x})$ yields the action

$$S[\mathbf{\Phi}] = \int \mathrm{d}^2x \int \mathrm{d}t \left\{ \frac{1}{2}(\partial_\mu \mathbf{\Phi}) \cdot (\partial^\mu \mathbf{\Phi}) + \lambda(t,\vec{x})(\mathbf{\Phi}\cdot\mathbf{\Phi} - 1) \right\}$$

Using $\delta S[\mathbf{\Phi}] = 0$ and $\mathbf{\Phi}\cdot\mathbf{\Phi} = 1$ we find the field equations

$$\Box\mathbf{\Phi} - (\mathbf{\Phi}\cdot\Box\mathbf{\Phi})\mathbf{\Phi} = 0 \quad \xrightarrow{\text{static solutions}} \quad \Delta\mathbf{\Phi} - (\mathbf{\Phi}\cdot\Delta\mathbf{\Phi})\mathbf{\Phi} = 0$$

A Legendre transform yields the energy functional

$$E = \frac{1}{2}\int \mathrm{d}^2x (\partial_k \mathbf{\Phi})^2 \equiv \frac{1}{2}\sum_{k=1}^{2}\sum_{i=1}^{3}\int \mathrm{d}^2x (\partial_k \Phi_i)^2 \quad \geq \quad 0$$

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- Ground state: $E = 0 \Leftrightarrow \mathbf{\Phi}(\vec{x}) \equiv \mathbf{\Phi}_0 = \text{const.}$, where $|\mathbf{\Phi}_0| = 1$
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The finite energy solutions ($0 < E < \infty$) are called solitons.
$\hookrightarrow$ To ensure $E < \infty$ we demand $\lim\limits_{|\vec{x}| \to \infty} \mathbf{\Phi}(\vec{x}) = \mathbf{\Phi}_0 = \text{const.}$

Definition: One-point compactification

$\mathbb{R}^2 \cup \{\infty\}$ with an extended topology is called one-point compactification of $\mathbb{R}^2$ and

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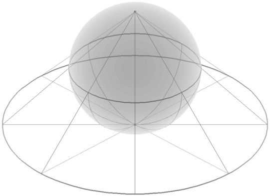

$\hookrightarrow$ [Wik12e]

In combination with the above boundary condition we may redefine $\mathbf{\Phi}$:

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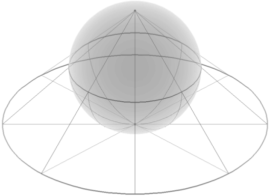

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Topological aspects: The Pontryagin number [Raj87, WZ83]

To obtain and classify the solutions we employ a topological invariant:

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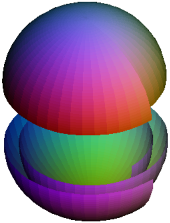
The Pontryagin number (topological charge) is

$$Q[\mathbf{\Phi}] := \frac{1}{8\pi} \int \mathrm{d}\vec{x}\, \varepsilon_{k,l} \mathbf{\Phi} \cdot (\partial_k \mathbf{\Phi} \times \partial_l \mathbf{\Phi})$$

$\widehat{=}$ how often $S^2_{(phy)}$ "wraps around" $S^2_{(int)}$



$\hookrightarrow$ [Wik12c]

Classification means: $\delta Q[\mathbf{\Phi}] = 0$ and $Q[\mathbf{\Phi}_1] = Q[\mathbf{\Phi}_2] \Leftrightarrow \mathbf{\Phi}_1 \simeq \mathbf{\Phi}_2$

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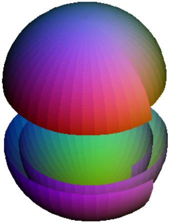

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A straightforward approach

- Wick rotation: $t \to \tau$, Minkowski metric $\to$ Euclidean metric
  $(t, \vec{x}) \in \mathbb{R}_{\text{Time}} \times \mathbb{R}^2_{\text{Space}} \to \vec{r} = (\tau, x, y) \in \mathbb{R}^3$
- Topological current: $\nabla \cdot \mathbf{J} = 0 \Rightarrow \exists \mathbf{A} : \mathbf{J} = \nabla \times \mathbf{A}$
- Coulomb-gauge: $\nabla \cdot \mathbf{A} = 0 \Rightarrow \nabla \times \mathbf{J} = \nabla \times (\nabla \times \mathbf{A}) = -\Delta \mathbf{A}$
- Solution for $\mathbf{A}$ (cf. Electrodynamics):

$$\mathbf{A}(\vec{r}) = \frac{1}{4\pi} \int \mathrm{d}^3 r' \frac{\nabla' \times \mathbf{J}(\vec{r}')}{|\vec{r} - \vec{r}\prime|} \overset{\text{I.b.P.}}{=} \frac{1}{4\pi} \int \mathrm{d}^3 r' \mathbf{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}\prime}{|\vec{r} - \vec{r}\prime|^3}$$

  $\hookrightarrow$ $\mathbf{A}[\mathbf{J}] = \mathbf{A}[\boldsymbol{\Phi}]$ is a non-local functional of $\boldsymbol{\Phi}$.
- Consider two skyrmions with world lines $\vec{\gamma}_{1,2}(\varphi)$ parametrized by φ.
  $\hookrightarrow$ $\mathbf{J}\,\mathrm{d}^3 r \approx J \cdot \mathrm{d}\vec{\gamma}_1 + J \cdot \mathrm{d}\vec{\gamma}_2$ for skyrmions with negligible spatial extent.
- It follows

$$\mathbf{A}(\vec{r}) = \frac{J}{4\pi} \int \mathrm{d}\vec{\gamma}_1 \times \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3} + \frac{J}{4\pi} \int \mathrm{d}\vec{\gamma}_2 \times \frac{\vec{r} - \vec{r}_2}{|\vec{r} - \vec{r}_2|^3}$$

A straightforward approach

- Since $\vec{\gamma}_i$ are closed curves, Stoke's theorem implies

$$\begin{aligned} \mathbf{A}(\vec{r}) &= \frac{J}{4\pi}\nabla\cdot\int_{\Sigma_1} \mathrm{d}\vec{n}_1 \times \frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|^3} + \frac{J}{4\pi}\nabla\cdot\int_{\Sigma_2} \mathrm{d}\vec{n}_2 \times \frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|^3} \\ &= \frac{J}{4\pi}\left[\nabla\Omega_1(\vec{r}) + \nabla\Omega_2(\vec{r})\right] \end{aligned}$$

  where $\Omega_i(\vec{r})$ denotes the solid angle of Σ_i viewed from $\vec{r}$.

- The Hopf invariant is

$$H = -\frac{1}{2\pi}\int \mathrm{d}^3 r\, \mathbf{A}(\vec{r})\mathbf{J}(\vec{r}) \approx -\frac{1}{2\pi}\sum_{i=1,2}\int_{\gamma_i} \mathrm{d}\vec{\gamma}_i\, \mathbf{A}(\vec{r}) = -\frac{J}{8\pi^2}\sum_{i,j=1,2}\int_{\gamma_i} \mathrm{d}\vec{\gamma}_i\, \nabla\Omega_j(\vec{r})$$

- Assume $\mathrm{d}\vec{\gamma}_i \cdot \nabla\Omega_i(\vec{r}) = 0$ for $i = 1, 2$ and set $J = \pi$:

$$\begin{aligned} H &= -\frac{J}{8\pi^2}\sum_{i\neq j}\int_{\gamma_1}\int_{\gamma_2} \frac{[(\vec{r}_i-\vec{r}_j)\times \mathrm{d}\vec{\gamma}_j]\cdot \mathrm{d}\vec{\gamma}_i}{|\vec{r}_i-\vec{r}_j|^3} = \frac{J}{4\pi^2}\int_{\gamma_1}\int_{\gamma_2} \frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|^3}\,(\mathrm{d}\vec{\gamma}_1\times \mathrm{d}\vec{\gamma}_2) \\ &= \mathrm{Link}[\gamma_1,\gamma_2] \quad\Rightarrow\quad \text{Linking number of world lines} \end{aligned}$$