Path integral formalism for bosonic fields

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Hauptseminar: Quantenfeldtheorie niedrigdimensionaler Systeme

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Agenda

1. Coherent state recap
2. Path integral for bosonic particles
3. Partition function with path integrals
4. Partition function for non interacting bosons
5. Thermal averages
6. Conclusion

Coherent state recap

- Coherent states $|\phi\rangle$ are eigenvectors of anniliation operators a_i

$$a_i |\phi\rangle = \phi_i |\phi\rangle$$

- Representation by number states:

$$|\phi\rangle = e^{\sum_\alpha \phi_\alpha a_\alpha^\dagger} |0\rangle$$

- Closure relation

$$\int \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} |0\rangle \langle 0|$$

- Trace of an Operator

$$trA = \int \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} \langle\phi| A |\phi\rangle$$

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Path integral for bosonic particles

Same procedure as used in the feynman path integral:

$$e^{-\frac{i}{\hbar}H\Delta t} = \underbrace{e^{-\frac{\epsilon}{\hbar}H}\dots e^{-\frac{\epsilon}{\hbar}H}}_{M\text{-times}}, \qquad \epsilon = \frac{\Delta t}{M}$$

Notation: $\phi_i = \phi_0, \quad \phi_f = \phi_M, \quad \phi_k = \{\phi_{\alpha,k}\}_{\alpha\in\mathbb{N}}, \quad a = \{a_\alpha\}_{\alpha\in\mathbb{N}}$

$$U(\phi_f, t_f, \phi_i, t_i) = \langle\phi_M| e^{-\frac{i\epsilon}{\hbar}H\Delta t} e^{-\frac{i\epsilon}{\hbar}H\Delta t} \dots e^{-\frac{i\epsilon}{\hbar}H\Delta t} |\phi_0\rangle$$

$$= \lim_{M\to\infty} \int \left(\prod_{k=1}^{M-1} \prod_\alpha \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i}\right) e^{-\sum_{k=1}^{M-1}\sum_\alpha \phi^*_{\alpha,k}\phi_{\alpha,k}}$$

$$\times \prod_{k=1}^{M} \langle\phi_k| e^{-\frac{i\epsilon}{\hbar}H(a^\dagger,a)} |\phi_{k-1}\rangle$$

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Path integral for bosonic particles

$$U(\phi_f, t_f, \phi_i, t_i) = \lim_{M\to\infty} \int \left(\prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \right) e^{-\sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k}\phi_{\alpha,k}} \times \prod_{k=1}^{M} \langle \phi_k | : e^{-\frac{i\epsilon}{\hbar} H(a^\dagger, a)} : + O(\epsilon^2) | \phi_{k-1} \rangle$$

$$= \lim_{M\to\infty} \int \left(\prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \right) \times e^{-\sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k}\phi_{\alpha,k-1} + \sum_{k=1}^{M} \left[\sum_{\alpha} \phi^*_{\alpha,k}\phi_{\alpha,k-1} - \frac{i\epsilon}{\hbar} H(\phi^*_k, \phi_{k-1}) \right]}$$

$$= \lim_{M\to\infty} \int \left(\prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \right) e^{S(M,\phi^*,\phi)}$$

Path integral for bosonic particles

$$U(\phi_f, t_f, \phi_i, t_i) = \lim_{M\to\infty} \int \left(\prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \right) e^{-\sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k}\phi_{\alpha,k}}$$

$$\times \prod_{k=1}^{M} \langle \phi_k | : e^{-\frac{i\epsilon}{\hbar} H(a^\dagger, a)} : + O(\epsilon^2) \, |\phi_{k-1}\rangle$$

$$= \lim_{M\to\infty} \int \left(\prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \right)$$

$$\times\, e^{-\sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k}\phi_{\alpha,k-1} + \sum_{k=1}^{M} \left[\sum_{\alpha} \phi^*_{\alpha,k}\phi_{\alpha,k-1} - \frac{i\epsilon}{\hbar} H(\phi^*_k, \phi_{k-1}) \right]}$$

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Path integral for bosonic particles

$$
\begin{aligned}
U(\phi_f, t_f, \phi_i, t_i) = \lim_{M\to\infty} \int \left(\prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \right) e^{-\sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k}\phi_{\alpha,k}} \\
\times \prod_{k=1}^{M} \langle \phi_k | : e^{-\frac{i\epsilon}{\hbar} H(a^\dagger, a)} : + O(\epsilon^2) \, | \phi_{k-1} \rangle \\
= \lim_{M\to\infty} \int \left(\prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \right) \\
\times e^{-\sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k}\phi_{\alpha,k-1} + \sum_{k=1}^{M} \left[\sum_{\alpha} \phi^*_{\alpha,k}\phi_{\alpha,k-1} - \frac{i\epsilon}{\hbar} H(\phi^*_k, \phi_{k-1}) \right]} \\
= \lim_{M\to\infty} \int \left(\prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \right) e^{S(M,\phi^*,\phi)}
\end{aligned}
$$

Path integral for bosonic particles

Merging sums

$$S(M,\phi^*,\phi) = \sum_\alpha \phi^*_{\alpha,M}\phi_{\alpha,M-1} - \frac{i\epsilon}{\hbar}H(\phi^*_M,\phi_{M-1}) + i\epsilon\sum_{k=1}^{M-1}\left[i\sum_\alpha \phi^*_{\alpha,k}\frac{(\phi_{\alpha,k}-\phi_{\alpha,k-1})}{\epsilon} - \frac{1}{\hbar}H(\phi^*_k,\phi_{k-1})\right]$$

Continuum notation:

$$\xrightarrow{M\to\infty} \sum_\alpha \phi^*_\alpha(t_f)\phi_\alpha(t_f) + \frac{i}{\hbar}\int_{t_i}^{t_f} dt \underbrace{\left[i\hbar\sum_\alpha \phi^*_\alpha(t)\frac{\partial\phi_\alpha(t)}{\partial t} - H(\phi^*(t),\phi(t))\right]}_{L(\phi^*(t),\phi(t))}$$

Path integral for bosonic particles

Merging sums

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Path integral for bosonic particles

Continuum Formulation

$$U(\phi_f, t_f, \phi_i, t_i) = \int\limits_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\left[\phi^*(t)\phi(t)\right] \exp\left(-\sum_\alpha \phi_\alpha(t_f)^* \phi_\alpha(t_f) \right)$$

$$\times \exp\left(\frac{i}{\hbar} \int\limits_{t_i}^{t_f} dt \left[i\hbar \sum_\alpha \phi_\alpha^*(t) \frac{\partial \phi_\alpha(t)}{\partial t} - H\left(\phi^*(t), \phi(t)\right) \right] \right)$$

Measure

$$D\left[\phi^*(t)\phi(t)\right] = \lim_{M\to\infty} \int \prod_{k=1}^{M-1} \prod_\alpha \frac{d\phi_{\alpha,k}^* d\phi_{\alpha,k}}{2\pi i}$$

Comparison $U(\xi_f, t_f, \xi_i, t_i)$

- Many particles, coherent states

$$\int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\big[\phi^*(t)\phi(t)\big] \exp\left(- \sum_\alpha \phi_\alpha(t_f)^* \phi_\alpha(t_f) \right)$$

$$\times \exp\left(\frac{i}{\hbar} \int_{t_i}^{t_f} dt \left[i\hbar \sum_\alpha \phi_\alpha^*(t) \frac{\partial \phi_\alpha(t)}{\partial t} - H\big(\phi^*(t), \phi(t)\big) \right] \right)$$

- Single particle, x, p representation

$$\int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}\left[p\right] \mathcal{D}\left[q\right] \exp\left(\frac{\mathrm{i}}{\hbar} \int_{t_i}^{t_f} \mathrm{d}t' \underbrace{p\dot{q} - H(p,q)}_{L} \right)$$

Partition function with path integrals

Grand canonical ensemble:

$$Z = \mathrm{tr}\left(e^{-\beta(H-\mu N)}\right)$$
$$= \int \left(\prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i}\right) e^{-\sum\limits_\alpha \phi_\alpha^* \phi_\alpha} \langle\phi| e^{-\beta(H-\mu N)} |\phi\rangle$$

New hamiltonian: $\hat{H} = H - \mu N$, substitution $t = -i\tau$

$$\Rightarrow U = e^{-\frac{i}{\hbar}\hat{H}\Delta t}$$
$$\Rightarrow \langle\phi| e^{-\beta(H-\mu N)} |\phi\rangle = U\left(\phi, -i\beta\hbar = \tau_f, \phi, 0 = \tau_i\right)$$

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Partition function with path integrals

Notation: $\epsilon = \frac{\beta}{M}$, $\qquad \phi_{\alpha,M} = \phi_{\alpha,0} = \phi_\alpha$, $\qquad N = \sum_\alpha a_\alpha^\dagger a_\alpha$

$$\langle\phi|\, e^{-\beta(H-\mu N)}\, |\phi\rangle = \lim_{M\to\infty} \int \left(\prod_{k=1}^{M-1} \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} \right) e^{-\sum_{k=1}^{M-1}\sum_\alpha \phi_{\alpha,k}^* \phi_{\alpha,k}} \times \prod_{k=1}^{M} \langle\phi_k|\, e^{-\epsilon\left(H(a^\dagger,a)-\mu N\right)}\, |\phi_{k-1}\rangle$$

$$\hookrightarrow \prod_{k=1}^{M} \langle\phi_k|\, : e^{-\epsilon\left(H(a^\dagger,a)-\mu N\right)} : + O(\epsilon^2)\, |\phi_{k-1}\rangle$$

$$\hookrightarrow e^{\sum_{k=1}^{M}\sum_\alpha \phi_{\alpha,k}^*\phi_{\alpha,k-1}}\, e^{-\epsilon \sum_{k=1}^{M}\left[H(\phi_k^*,\phi_{k-1}) - \mu\sum_\alpha \phi_{\alpha,k}^*\phi_{\alpha,k-1}\right]}$$

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$$\times \prod_{k=1}^{M} \langle\phi_k| e^{-\epsilon\left(H(a^\dagger,a)-\mu N\right)} |\phi_{k-1}\rangle$$

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$$\hookrightarrow e^{\sum_{k=1}^{M} \sum_\alpha \phi_{\alpha,k}^* \phi_{\alpha,k-1}} e^{-\epsilon \sum_{k=1}^{M} \left[H(\phi_k^*, \phi_{k-1}) - \mu \sum_\alpha \phi_{\alpha,k}^* \phi_{\alpha,k-1} \right]}$$

Partition function with path integrals

$$Z = \lim_{M\to\infty}\int\left(\prod_{k=1}^{M}\prod_{\alpha}\frac{d\phi^*_\alpha d\phi_\alpha}{2\pi i}\right)e^{-\sum_{k=1}^{M}\sum_\alpha \phi^*_{\alpha,k}\phi_{\alpha,k}}$$

$$\times\, e^{-\epsilon\sum_{k=1}^{M}\left(H(\phi^*_k,\phi_{k-1})-(1+\mu)\sum_\alpha \phi^*_{\alpha,k}\phi_{\alpha,k-1}\right)}$$

$$-\sum_{k=1}^{M}\sum_\alpha \phi^*_{\alpha,k}\phi_{\alpha,k} - \sum_{k=1}^{M}\left(\epsilon H(\phi^*_k,\phi_{k-1}) - (1+\epsilon\mu)\sum_\alpha \phi^*_{\alpha,k}\phi_{\alpha,k-1}\right)$$

$$= -\epsilon\sum_{k=1}^{M}\left[H(\phi^*_k,\phi_{k-1}) + \sum_\alpha \phi^*_{\alpha,k}\left(\frac{\phi_{\alpha,k}-\phi_{\alpha,k-1}}{\epsilon} - \mu\phi_{\alpha,k-1}\right)\right]$$

$$\xrightarrow{M\to\infty} -\int_0^\beta d\tau\left[H(\phi^*(\tau),\phi(\tau)) + \sum_\alpha \phi^*_\alpha(\tau)(\partial_\tau-\mu)\phi_\alpha(\tau)\right]$$

Partition function with path integrals

$$Z = \lim_{M\to\infty} \int \left(\prod_{k=1}^{M} \prod_{\alpha} \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} \right) e^{-\sum_{k=1}^{M} \sum_\alpha \phi_{\alpha,k}^* \phi_{\alpha,k}} \times e^{-\epsilon \sum_{k=1}^{M} \left(H(\phi_k^*, \phi_{k-1}) - (1+\mu) \sum_\alpha \phi_{\alpha,k}^* \phi_{\alpha,k-1} \right)}$$

$$- \sum_{k=1}^{M} \sum_{\alpha} \phi_{\alpha,k}^* \phi_{\alpha,k} - \sum_{k=1}^{M} \left(\epsilon H(\phi_k^*, \phi_{k-1}) - (1+\epsilon\mu) \sum_\alpha \phi_{\alpha,k}^* \phi_{\alpha,k-1} \right)$$

$$= -\epsilon \sum_{k=1}^{M} \left[H(\phi_k^*, \phi_{k-1}) + \sum_\alpha \phi_{\alpha,k}^* \left(\frac{\phi_{\alpha,k} - \phi_{\alpha,k-1}}{\epsilon} - \mu \phi_{\alpha,k-1} \right) \right]$$

$$\xrightarrow{M\to\infty} - \int_0^\beta d\tau \left[H(\phi^*(\tau), \phi(\tau)) + \sum_\alpha \phi_\alpha^*(\tau) (\partial_\tau - \mu) \phi_\alpha(\tau) \right]$$

Partition function with path integrals

$$Z = \lim_{M\to\infty} \int \left(\prod_{k=1}^{M} \prod_{\alpha} \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} \right) e^{-\sum\limits_{k=1}^{M} \sum\limits_{\alpha} \phi_{\alpha,k}^* \phi_{\alpha,k}}$$
$$\times\, e^{-\epsilon \sum\limits_{k=1}^{M} \left(H(\phi_k^*, \phi_{k-1}) - (1+\mu) \sum\limits_{\alpha} \phi_{\alpha,k}^* \phi_{\alpha,k-1} \right)}$$

$$-\sum_{k=1}^{M} \sum_{\alpha} \phi_{\alpha,k}^* \phi_{\alpha,k} - \sum_{k=1}^{M} \left(\epsilon H(\phi_k^*, \phi_{k-1}) - (1+\epsilon\mu) \sum_{\alpha} \phi_{\alpha,k}^* \phi_{\alpha,k-1} \right)$$
$$= -\epsilon \sum_{k=1}^{M} \left[H(\phi_k^*, \phi_{k-1}) + \sum_{\alpha} \phi_{\alpha,k}^* \left(\frac{\phi_{\alpha,k} - \phi_{\alpha,k-1}}{\epsilon} - \mu \phi_{\alpha,k-1} \right) \right]$$
$$\xrightarrow{M\to\infty} -\int_0^\beta d\tau \left[H\big(\phi^*(\tau), \phi(\tau)\big) + \sum_{\alpha} \phi_\alpha^*(\tau) \big(\partial_\tau - \mu\big) \phi_\alpha(\tau) \right]$$

Partition function with path integrals

Discrete and continous notation

$$Z = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} \times e^{-\epsilon \sum_{k=1}^{M} \left[H(\phi_k^*, \phi_{k-1}) + \sum_\alpha \phi_{\alpha,k}^* \left(\frac{\phi_{\alpha,k} - \phi_{\alpha,k-1}}{\epsilon} - \mu \phi_{\alpha,k-1} \right) \right]}$$

$$= \int_{\phi_\alpha(\beta)=\phi_\alpha(0)} D\left[\phi^*(\tau)\phi(\tau)\right] e^{-\int_0^\beta d\tau \left[H\left(\phi^*(\tau), \phi(\tau)\right) + \sum_\alpha \phi_\alpha^*(\tau) \left(\partial_\tau - \mu\right) \phi_\alpha(\tau) \right]}$$

Minkowski and Euclidean action

Substitution $t = -i\tau \quad \Rightarrow \quad (\Delta t)^2 \to -(\Delta\tau)^2$

$$Z = \int\limits_{\phi_\alpha(\beta)=\phi_\alpha(0)} D\Big[\phi^*(\tau)\phi(\tau)\Big] e^{-\int\limits_0^\beta d\tau \left[\hat{H}\left(\phi^*(\tau),\phi(\tau)\right)+\sum\limits_\alpha \phi^*_\alpha(\tau)\frac{\partial\phi_\alpha(\tau)}{\partial\tau}\right]}$$

$$U(...) = \int\limits_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\Big[\phi^*(t)\phi(t)\Big] e^{-\sum\limits_\alpha \phi_\alpha(t_f)^*\phi_\alpha(t_f)}$$

$$\times\, e^{\frac{i}{\hbar}\int\limits_{t_i}^{t_f} dt \left[i\hbar\sum\limits_\alpha \phi^*_\alpha(t)\frac{\partial\phi_\alpha(t)}{\partial t} - H(\phi^*(t),\phi(t))\right]}$$

Partition function of non interacting bosons

Hamiltonian:

$$H = \sum_{\alpha} \epsilon_\alpha a^\dagger_\alpha a_\alpha$$

Grand canonical ensemble

$$Z = \lim_{M\to\infty} \prod_\alpha \int \prod_{k=1}^{M} \frac{d\phi^*_\alpha d\phi_\alpha}{2\pi i}$$

$$e^{-\frac{\beta}{M}\sum_{k=1}^{M}\left[\epsilon_\alpha \phi^*_{\alpha,k}\phi_{\alpha,k-1} + \phi^*_{\alpha,k}\left(M\frac{\phi_{\alpha,k}-\phi_{\alpha,k-1}}{\beta} - \mu\phi_{\alpha,k-1}\right)\right]}$$

$$\hookrightarrow e^{-\sum_{k=1}^{M}\left[\left(\frac{\beta}{M}(\epsilon_\alpha-\mu)-1\right)\phi^*_{\alpha,k}\phi_{\alpha,k-1} + \phi^*_{\alpha,k}\phi_{\alpha,k}\right]}$$

Partition function of non interacting bosons

Hamiltonian:

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$$\hookrightarrow e^{-\sum_{k=1}^{M}\left[\left(\frac{\beta}{M}(\epsilon_\alpha-\mu)-1\right)\phi_{\alpha,k}^*\phi_{\alpha,k-1} + \phi_{\alpha,k}^*\phi_{\alpha,k}\right]}$$

$$-\sum_{k=1}^{M}\left[\left(\frac{\beta}{M}(\epsilon_\alpha-\mu)-1\right)\phi^*_{\alpha,k}\phi_{\alpha,k-1}+\phi^*_{\alpha,k}\phi_{\alpha,k}\right]=-\boldsymbol{\phi}^*_{\boldsymbol{\alpha}}\boldsymbol{S}^{(\boldsymbol{\alpha})}\boldsymbol{\phi}_{\boldsymbol{\alpha}}$$

$$\boldsymbol{\phi}_{\boldsymbol{\alpha}}=\begin{pmatrix}\phi_{\alpha,1}\\ \phi_{\alpha,2}\\ \vdots\\ \\ \vdots\\ \phi_{\alpha,M}\end{pmatrix},\qquad \boldsymbol{S}^{(\boldsymbol{\alpha})}=\begin{bmatrix}1 & 0 & \cdots & & \cdots 0 & -a\\ -a & 1 & 0 & & & 0\\ 0 & -a & 1 & & & \vdots\\ \vdots & 0 & \ddots & \ddots & 0 & \vdots\\ \vdots & & & -a & 1 & 0\\ 0 & \cdots & \cdots & 0 & -a & 1\end{bmatrix}$$

$$a=1-\frac{\beta}{M}(\epsilon_\alpha-\mu)$$

Multidimensional gaussian integral

$$\int \prod_{k=1}^{M} \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\boldsymbol{\phi_\alpha^* S^{(\alpha)} \phi_\alpha}} = \int \prod_{k=1}^{M} \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\boldsymbol{\phi_\alpha^* U U^\dagger S^{(\alpha)} U U^\dagger \phi_\alpha}}$$

$$= \int \prod_{k=1}^{M} \frac{d\tilde{\phi}_\alpha^* d\tilde{\phi}_\alpha}{2\pi i} e^{-\boldsymbol{\tilde{\phi}_\alpha^* D^{(\alpha)} \tilde{\phi}_\alpha}}$$

$$= \int \prod_{k=1}^{M} \frac{d\tilde{\phi}_\alpha^* d\tilde{\phi}_\alpha}{2\pi i} e^{-\sum_{k=1}^{M} d_k \phi_{\alpha,k}^* \phi_{\alpha,k}}$$

Change of variables: $\phi_\alpha = \phi_{\alpha,1} + i\phi_{\alpha,2}$, jacobian determant equals $2i$

$$= \int \prod_{k=1}^{M} \frac{d\tilde{\phi}_{\alpha,1} d\tilde{\phi}_{\alpha,2}}{\pi} e^{-\sum_{k=1}^{M} d_k \left(\phi_{\alpha,1}^2 + \phi_{\alpha,2}^2\right)} = \frac{1}{\pi^M} \prod_{k=1}^{M} \sqrt{\frac{\pi}{d_k}} \sqrt{\frac{\pi}{d_k}} = \frac{1}{\det \boldsymbol{S}^{\boldsymbol{\alpha}}}$$

Multidimensional gaussian integral

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Determinant

$$\det \boldsymbol{S}^{\boldsymbol{\alpha}} = \sum_P \varepsilon_{i_1,i_2,\ldots,i_M} S^\alpha_{1,i_1} S^\alpha_{2,i_2} \cdots S^\alpha_{M,i_M}$$
$$= 1 + \underbrace{\varepsilon_{2,3,4,\ldots M,1}}_{(-1)^{M-1}} (-a)^M$$

$$\begin{bmatrix} 1 & 0 & \cdots & & \cdots 0 & -a \\ -a & 1 & 0 & & & 0 \\ 0 & -a & 1 & & & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & & & -a & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -a & 1 \end{bmatrix}$$

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$$= 1 + \underbrace{\varepsilon_{2,3,4,\ldots M,1}}_{(-1)^{M-1}} (-a)^M = 1 - a^M = 1 - \left(1 - \frac{\beta}{M}(\epsilon_\alpha - \mu)\right)^M$$

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$$\to 1 - e^{-\beta(\epsilon_\alpha - \mu)}$$

$$\begin{bmatrix} 1 & 0 & \cdots & & \cdots 0 & -a \\ -a & 1 & 0 & & & 0 \\ 0 & -a & 1 & & & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & & & -a & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -a & 1 \end{bmatrix}$$

Result

$$Z = \lim_{M\to\infty} \prod_\alpha \frac{1}{|\boldsymbol{S_\alpha}|}$$
$$= \prod_\alpha \frac{1}{1 - e^{-\beta(\epsilon_\alpha - \mu)}}$$

Now: Use thermodynamik relations: $Z = e^{-\beta\Omega}$, $\frac{\partial\Omega}{\partial\mu} = -N$

$$\langle N \rangle = \sum_\alpha \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1}$$

Or... do it the hard way...

Thermal averages

Thermal average of an operator A:

$$\langle A \rangle = \frac{\sum_\alpha \langle \psi_\alpha | e^{-\beta(H-\mu N)} A | \psi_\alpha \rangle}{\sum_\alpha \langle \psi_\alpha | e^{-\beta(H-\mu N)} | \psi_\alpha \rangle}$$

With $N_\gamma = a_\gamma^\dagger a_\gamma$, $\Rightarrow N = \sum_\alpha N_\alpha$

$$\langle N_\gamma \rangle = \frac{1}{Z} \prod_\alpha \int \frac{d\phi^*_{\alpha,M} d\phi_{\alpha,M}}{2\pi i} \frac{d\phi^*_{\alpha,0} d\phi_{\alpha,0}}{2\pi i} e^{-\phi^*_{\alpha,M}\phi_{\alpha,M} - \phi^*_{\alpha,0}\phi_{\alpha,0}}$$

$$\langle \phi_M | e^{-\beta(H-\mu N)} | \phi_0 \rangle \underbrace{\langle \phi_0 | a_\gamma^\dagger a_\gamma | \phi_M \rangle}_{\phi^*_{\gamma,0}\phi_{\gamma,M} e^{\phi^*_{\alpha,0}\phi_{\alpha,M}}}$$

Thermal averages

Thermal average of an operator A:

$$\langle A\rangle = \frac{\sum_\alpha \langle\psi_\alpha|\, e^{-\beta(H-\mu N)} A\, |\psi_\alpha\rangle}{\sum_\alpha \langle\psi_\alpha|\, e^{-\beta(H-\mu N)}\, |\psi_\alpha\rangle}$$

With $N_\gamma = a^\dagger_\gamma a_\gamma$, $\Rightarrow N = \sum_\alpha N_\alpha$

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$$\langle\phi_M|\, e^{-\beta(H-\mu N)}\, |\phi_0\rangle \;\underbrace{\langle\phi_0|\, a^\dagger_\gamma a_\gamma\, |\phi_M\rangle}_{\phi^*_{\gamma,0}\phi_{\gamma,M} e^{\phi^*_{\alpha,0}\phi_{\alpha,M}}}$$

Thermal averages

$$\langle N_\gamma \rangle = \frac{1}{Z} \lim_{M\to\infty} \prod_\alpha \int \prod_{k=0}^{M} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} e^{-\phi^*_{\alpha,M}\phi_{\alpha,M} - \phi^*_{\alpha,0}\phi_{\alpha,0} + \phi^*_{\alpha,0}\phi_{\alpha,M}}$$

$$\phi^*_{\gamma,0}\phi_{\gamma,M} \prod_{k=1}^{M} \langle \phi_k | : e^{-\beta(H-\mu N)} : + O(\epsilon)^2 | \phi_{k-1} \rangle$$

$$= \frac{1}{Z} \lim_{M\to\infty} \prod_\alpha \int \prod_{k=0}^{M} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \phi^*_{\gamma,0}\phi_{\gamma,M} e^{\boldsymbol{\phi^*_\alpha S^\alpha \phi_\alpha}}$$

$$= \lim_{M\to\infty} \frac{\int \prod_{k=0}^{M} \frac{d\phi^*_{\gamma,k} d\phi_{\gamma,k}}{2\pi i} \phi^*_{\gamma,0}\phi_{\gamma,M} e^{\boldsymbol{\phi^*_\gamma S^\gamma \phi_\gamma}}}{\int \prod_{k=1}^{M} \frac{d\phi^*_{\gamma,k} d\phi_{\gamma,k}}{2\pi i} e^{\boldsymbol{\phi^*_\gamma S^\gamma \phi_\gamma}}}$$

Thermal averages

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$$= \lim_{M\to\infty} \frac{\partial^2}{\partial J^*_M \partial J_0} \frac{\int \prod_{k=0}^{M} \frac{d\phi^*_{\gamma,k} d\phi_{\gamma,k}}{2\pi i} e^{\boldsymbol{\phi^*_\gamma S^\gamma \phi_\gamma + J^* \phi_\gamma + \phi^*_\gamma J}}}{\int \prod_{k=1}^{M} \frac{d\phi^*_{\gamma,k} d\phi_{\gamma,k}}{2\pi i} e^{\boldsymbol{\phi^*_\gamma S^\gamma \phi_\gamma}}} \Bigg|_{\boldsymbol{J^*=J=0}}$$

$$= \lim_{M\to\infty} \frac{\partial^2}{\partial J^*_M \partial J_0} |\boldsymbol{S^\gamma}| \frac{1}{|\boldsymbol{S^\gamma}|} e^{\boldsymbol{J^*}\left(\boldsymbol{S^\gamma}\right)^{-1}\boldsymbol{J}} \Bigg|_{\boldsymbol{J^*=J=0}} = \lim_{M\to\infty} \left(\boldsymbol{S^\gamma}\right)^{-1}_{0,M}$$

Thermal averages

$$\langle N_\gamma \rangle = \lim_{M\to\infty} \frac{\int \prod_{k=0}^{M} \frac{d\phi^*_{\gamma,k} d\phi_{\gamma,k}}{2\pi i} \phi^*_{\gamma,0}\phi_{\gamma,M} e^{\boldsymbol{\phi^*_\gamma S^\gamma \phi_\gamma}}}{\int \prod_{k=1}^{M} \frac{d\phi^*_{\gamma,k} d\phi_{\gamma,k}}{2\pi i} e^{\boldsymbol{\phi^*_\gamma S^\gamma \phi_\gamma}}}$$

$$= \lim_{M\to\infty} \frac{\partial^2}{\partial J^*_M \partial J_0} \frac{\int \prod_{k=0}^{M} \frac{d\phi^*_{\gamma,k} d\phi_{\gamma,k}}{2\pi i} e^{\boldsymbol{\phi^*_\gamma S^\gamma \phi_\gamma + J^* \phi_\gamma + \phi^*_\gamma J}}}{\int \prod_{k=1}^{M} \frac{d\phi^*_{\gamma,k} d\phi_{\gamma,k}}{2\pi i} e^{\boldsymbol{\phi^*_\gamma S^\gamma \phi_\gamma}}} \Bigg|_{\boldsymbol{J^*=J=0}}$$

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Thermal averages

$$a_\alpha = 1 - \frac{\beta}{M}\left(\epsilon_\alpha - \mu\right)$$

$$\begin{aligned}\langle N_\gamma \rangle &= \lim_{M\to\infty} \left(\boldsymbol{S}^{\boldsymbol{\gamma}}\right)^{-1}_{0,M} = \lim_{M\to\infty} \frac{a_\gamma^{M-1}}{1 - a_\gamma^M} = \lim_{M\to\infty} \frac{1}{a_\gamma^{1-M} - a_\gamma} \\ &= \lim_{M\to\infty} \frac{1}{\left[1 - \frac{\beta}{M}\left(\epsilon_\alpha - \mu\right)\right]^{M-1} - 1} = \frac{1}{e^{\beta(H-\mu)} - 1} = n_\gamma\end{aligned}$$

Conclusion

Take home message

- Coherent states are suitable to develop a path integral formalism for many particle systems.

$$U(\phi_f, t_f, \phi_i, t_i) = \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\big[\phi^*(t)\phi(t)\big] e^{-\sum_\alpha \phi_\alpha(t_f)^* \phi_\alpha(t_f)} \times e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[i\hbar \sum_\alpha \phi_\alpha^*(t) \frac{\partial \phi_\alpha(t)}{\partial t} - H\left(\phi^*(t), \phi(t)\right)\right]}$$

- A complex time enables us to express the partition function with the time evolution operator.

$$Z = \int_{\phi_\alpha(\beta)=\phi_\alpha(0)} D\big[\phi^*(\tau)\phi(\tau)\big] e^{-\int_0^\beta d\tau \left[H\left(\phi^*(\tau), \phi(\tau)\right) + \sum_\alpha \phi_\alpha^*(\tau)\left(\partial_\tau - \mu\right)\phi_\alpha(\tau)\right]}$$

Conclusion

Take home message

- Coherent states are suitable to develop a path integral formalism for many particle systems.

$$U(\phi_f, t_f, \phi_i, t_i) = \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\left[\phi^*(t)\phi(t)\right] e^{-\sum_\alpha \phi_\alpha(t_f)^* \phi_\alpha(t_f)} \times e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[i\hbar \sum_\alpha \phi_\alpha^*(t) \frac{\partial \phi_\alpha(t)}{\partial t} - H\left(\phi^*(t), \phi(t)\right)\right]}$$

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