Feynman path integral formalism

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Hauptseminar: Quantenfeldtheorie niedrigdimensionaler Systeme

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Agenda

1. From Schrödinger to Feynman
2. Hands-on example: Free particle
3. Semiclassical approach
4. Harmonic Oscillator
5. Connection to statistical physics
6. Conclusion

Time evolution

Schrödinger equation

$$\mathrm{i}\hbar\partial_t |\psi\rangle = H |\psi\rangle$$

Formal solution (time independent H)

$$|\psi(t)\rangle = \underbrace{\exp\left(\frac{tH}{\mathrm{i}\hbar}\right)}_{U(t)} |\psi(0)\rangle$$

Matrix elements $\langle q_f| \exp\left(\frac{tH}{\mathrm{i}\hbar}\right) |q_i\rangle$ Propagation $(q_i, 0) \to (q_f, t)$

Time evolution

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Formal solution (time independent H)

$$|\psi(t)\rangle = \underbrace{\exp\left(\frac{tH}{\mathrm{i}\hbar}\right)}_{U(t)} \int \mathrm{d}q\, |q\rangle \langle q| \, |\psi(0)\rangle$$

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Schrödinger equation

$$\mathrm{i}\hbar\partial_t |\psi\rangle = H |\psi\rangle$$

Formal solution (time independent H) position representation

$$\langle q'| |\psi(t)\rangle = \langle q'| \underbrace{\exp\left(\frac{tH}{\mathrm{i}\hbar}\right)}_{U(t)} \int \mathrm{d}q\, |q\rangle \langle q| |\psi(0)\rangle$$

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Time evolution

Schrödinger equation

$$\mathrm{i}\hbar\partial_t \left|\psi\right\rangle = H \left|\psi\right\rangle$$

Formal solution (time independent H) position representation

$$\left\langle q'\right| \left|\psi(t)\right\rangle = \left\langle q'\right| \underbrace{\exp\left(\frac{tH}{\mathrm{i}\hbar}\right)}_{U(t)} \int \mathrm{d}q \left|q\right\rangle \left\langle q\right| \left|\psi(0)\right\rangle$$

Matrix elements $\left\langle q_f\right| \exp\left(\frac{tH}{\mathrm{i}\hbar}\right) \left|q_i\right\rangle$ Propagation $(q_i, 0) \to (q_f, t)$

Small steps

- Hard/impossible to solve
- Small steps; $\Delta t = t/N$

$$\exp\left(\frac{tH}{\mathrm{i}\hbar}\right) = \left[\exp\left(\frac{\Delta t H}{\mathrm{i}\hbar}\right)\right]^N$$

- Ordering: p left of q

$$H = T(p) + V(q) \Rightarrow \mathrm{e}^{\frac{\Delta t H}{\mathrm{i}\hbar}} = \mathrm{e}^{\frac{\Delta t T}{\mathrm{i}\hbar}}\,\mathrm{e}^{\frac{\Delta t V}{\mathrm{i}\hbar}} + O(\Delta t^2)$$
$$\Leftrightarrow \mathrm{e}^{\frac{\Delta t H}{\mathrm{i}\hbar}} =: \mathrm{e}^{\frac{\Delta t H}{\mathrm{i}\hbar}} : + O(\Delta t^2)$$

$$\frac{\Delta t H}{\mathrm{i}\hbar} = T + V =: \frac{\Delta t H}{\mathrm{i}\hbar} :$$

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$$\left(\frac{\Delta t H}{\mathrm{i}\hbar}\right)^2 \neq : \left(\frac{\Delta t H}{\mathrm{i}\hbar}\right)^2 :$$

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$$\Leftrightarrow \mathrm{e}^{\frac{\Delta t H}{\mathrm{i}\hbar}} =: \mathrm{e}^{\frac{\Delta t H}{\mathrm{i}\hbar}} : + O(\Delta t^2)$$

$$T^2 + TV + VT + V^2 \neq T^2 + 2TV + V^2 \quad \text{if } [T, V] \neq 0$$

Small steps

$$\langle q_f | \exp\left(\frac{tH}{\mathrm{i}\hbar}\right) | q_i \rangle \simeq \langle q_f | :\exp\left(\frac{tH}{\mathrm{i}\hbar}\right): | q_i \rangle$$

$$= \langle q_f | \; :\exp\left(\frac{\Delta t H}{\mathrm{i}\hbar}\right): \; :\exp\left(\frac{\Delta t H}{\mathrm{i}\hbar}\right): \; \cdots \; :\exp\left(\frac{\Delta t H}{\mathrm{i}\hbar}\right): | q_i \rangle$$

$$1 = \int \mathrm{d}q_n \int \mathrm{d}p_n \, |q_n\rangle \langle q_n | p_n \rangle \langle p_n | \quad n = 1, \dots, N \text{ with } q_0 = q_i, q_N = q_f$$

Ordering pays off:

$$\langle p_n | \exp\left(\frac{\Delta t \hat{T}}{\mathrm{i}\hbar}\right) = \exp\left(\frac{\Delta t T(p_n)}{\mathrm{i}\hbar}\right) \langle p_n |$$

$$\exp\left(\frac{\Delta t \hat{V}}{\mathrm{i}\hbar}\right) | q_n \rangle = | q_n \rangle \exp\left(\frac{\Delta t V(q_n)}{\mathrm{i}\hbar}\right)$$

$$\langle q | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \mathrm{e}^{\frac{\mathrm{i}pq}{\hbar}} = \frac{1}{\sqrt{2\pi\hbar}} \mathrm{e}^{\frac{-pq}{\mathrm{i}\hbar}}$$

Small steps

$$\langle q_f|\exp\left(\frac{tH}{\mathrm{i}\hbar}\right)|q_i\rangle \simeq \langle q_f|:\exp\left(\frac{tH}{\mathrm{i}\hbar}\right):|q_i\rangle$$

$$= \langle q_f|\,1:\exp\left(\frac{\Delta tH}{\mathrm{i}\hbar}\right):1:\exp\left(\frac{\Delta tH}{\mathrm{i}\hbar}\right):1\cdots 1:\exp\left(\frac{\Delta tH}{\mathrm{i}\hbar}\right):|q_i\rangle$$

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$$\exp\left(\frac{\Delta t\hat{V}}{\mathrm{i}\hbar}\right)|q_n\rangle = |q_n\rangle\exp\left(\frac{\Delta tV(q_n)}{\mathrm{i}\hbar}\right)$$

$$\langle q|\,p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{\frac{ipq}{\hbar}} = \frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{\frac{-pq}{i\hbar}}$$

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$$= \langle q_f | \, 1 : \exp\left(\frac{\Delta t H}{\mathrm{i}\hbar}\right) : 1 : \exp\left(\frac{\Delta t H}{\mathrm{i}\hbar}\right) : 1 \cdots 1 : \exp\left(\frac{\Delta t H}{\mathrm{i}\hbar}\right) : |q_i\rangle$$

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Small steps

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$$\mathbf{1} = \int \mathrm{d}q_n \int \mathrm{d}p_n \, |q_n\rangle \langle q_n|\, p_n\rangle \langle p_n| \quad n = 1, \ldots, N \text{ with } q_0 = q_i, q_N = q_f$$

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Path integral

$$\langle q_f | \exp\left(\frac{tH}{\mathrm{i}\hbar}\right) | q_i \rangle \simeq \int \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1} \int \frac{\mathrm{d}p_1}{2\pi\hbar} \cdots \frac{\mathrm{d}p_N}{2\pi\hbar}$$
$$\times \exp\left(\frac{\Delta t}{\mathrm{i}\hbar} \sum_{n=0}^{N-1} \left[V(q_n) + T(p_{n+1}) - p_{n+1} \frac{q_{n+1} - q_n}{\Delta t} \right]\right)$$

$N \to \infty$: $\{q_n\} \to q(t')$, $\Delta t \sum \to \int \mathrm{d}t'$, $\frac{\Delta q}{\Delta t} \to \dot{q}$, $T(p_{n+1}) \to T(p(t'))$, ...

$$V + T - p\dot{q} = H - p\dot{q} = (p\dot{q} - L) - p\dot{q}$$

Hamilton formulation of the path integral

$$\int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}[p]\, \mathcal{D}[q] \exp\left(\frac{\mathrm{i}}{\hbar} \int_0^t \mathrm{d}t' \underbrace{p\dot{q} - H(p,q)}_{L}\right)$$

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$$\times \exp\left(\frac{\Delta t}{\mathrm{i}\hbar}\sum_{n=0}^{N-1}\left[V(q_n)+T(p_{n+1})-p_{n+1}\frac{q_{n+1}-q_n}{\Delta t}\right]\right)$$

$N\to\infty$: $\{q_n\}\to q(t')$, $\Delta t\sum\to\int \mathrm{d}t'$, $\frac{\Delta q}{\Delta t}\to\dot{q}$, $T(p_{n+1})\to T(p(t'))$, ...

$$V+T-p\dot{q}=H-p\dot{q}=(p\dot{q}-L)-p\dot{q}$$

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- Notation
- Fast fluctuation of phase ⇒ cancellation

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Hamilton formulation of the path integral

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Kinetic energy quadratic in p

$$\int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}\left[q\right] \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t' V(q)\right)$$

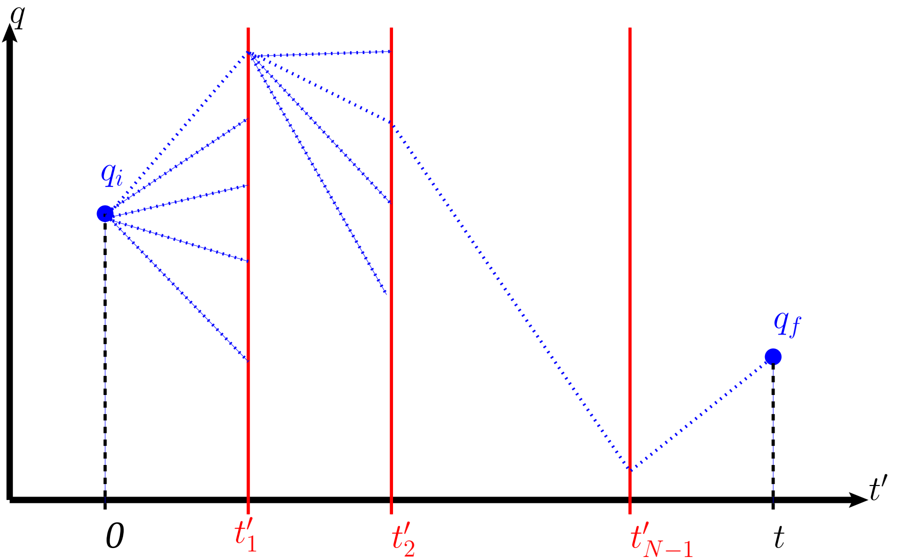
$$\times \int \underbrace{\frac{\mathrm{d}p_1}{2\pi\hbar}\cdots\frac{\mathrm{d}p_N}{2\pi\hbar}}_{\mathcal{D}[p]} \exp\left(-\frac{\mathrm{i}\Delta t}{\hbar}\sum_{n=1}^{N} \underbrace{\frac{p_n^2}{2m} - p_n \frac{q_n - q_{n-1}}{\Delta t}}_{\frac{1}{2m}(p_n - m\frac{\Delta q}{\Delta t})^2 - \frac{1}{2}m\left(\frac{\Delta q}{\Delta t}\right)^2}\right)$$

$$\int \mathrm{d}x \, \mathrm{e}^{\mathrm{i}x^2} = \sqrt{\mathrm{i}\pi} \qquad \text{(Fresnel)}$$

Lagrangian form of the path integral

$$\int_{q(0)=q_i}^{q(t)=q_f} \tilde{\mathcal{D}}\left[q\right] \exp\left(\frac{\mathrm{i}}{\hbar}S[q]\right), \; \tilde{\mathcal{D}}\left[q\right] = \lim_{N\to\infty}\left(\frac{Nm}{\mathrm{i}t2\pi\hbar}\right)^{N/2} \mathrm{d}q_1 \cdots \mathrm{d}q_{N-1}$$

Path integral – intuition



Free particle

$$H = \frac{p^2}{2m}$$

$$\vdots$$

$$\langle q_f | \exp\left(\frac{t}{\mathrm{i}\hbar}\frac{p^2}{2m}\right)|q_i\rangle = \left(\frac{m}{2\pi\mathrm{i}\hbar t}\right)^{1/2} \exp\left(\frac{\mathrm{i}}{\hbar}\frac{m}{2t}(q_f - q_i)^2\right)$$

Semiclassical approach – fast oscillations

$$\langle q_f | U(t) | q_i \rangle = \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}[q] \exp\left(\frac{\mathrm{i}}{\hbar} S[q]\right)$$

- Classical physics: $\hbar \to 0$
- Change in path $q \Rightarrow$ fast oscillations $\Rightarrow$ cancellations
- Exception: Stationary action: $\delta S = 0 \Rightarrow$ constructive interference

Stationary phase approximation

$$\int \mathrm{d}x \mathrm{e}^{-f(x)} = \int \mathrm{d}r \mathrm{e}^{-f(\xi+r)} \approx \int \mathrm{d}r \mathrm{e}^{-f(\xi)-\frac{1}{2}f''(\xi)r^2}$$

$$\int \mathcal{D}[x]\, \mathrm{e}^{-F[x]} = \int \mathcal{D}[r]\, \mathrm{e}^{-F[\xi+r]} \approx \int \mathcal{D}[r]\, \mathrm{e}^{-F[\xi]-\frac{1}{2}\delta^2 F[\xi] r^2}$$

$$\int \mathcal{D}[r]\, \mathrm{e}^{-F[\xi]-\frac{1}{2}\int \mathrm{d}t' \int \mathrm{d}t r(t') \left.\frac{\delta^2 F}{\delta x(t)\delta x(t')}\right|_{\xi} r(t)} = \mathrm{e}^{-F[\xi]} \det\left(\frac{1}{2\pi}\frac{\delta^2 F}{\delta x(t)\delta x(t')}\right)^{-1/2}$$

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Harmonic oscillator

$$\langle q_f | U(t) | q_i \rangle = \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}[q] \exp\left(\frac{\mathrm{i}}{\hbar} \int_0^t \mathrm{d}t' \frac{1}{2} m\dot{q}^2 - \frac{1}{2} m\omega^2 q^2 \right)$$

Stationary phase ~~approximation~~ computation

- Classical paths: $q_c(t') = A \sin \omega t' + B \cos \omega t' \quad \to S[q_c]$.
- Vicinity of classical path:
  $S[q_c + r] = S[q_c] + \int_0^t \mathrm{d}t' \, r(t') \underbrace{[-\frac{m}{2}][\partial_{t'}^2 + \omega^2]}_{} r(t')$
- Gaussian functional integral: $\det A = \prod_i a_i$ (with $A\phi_i = a_i \phi_i$)
  $\phi_i(t') = \sin(n\pi t'/t)$ and $a_i = \frac{m}{2} \left[n^2\pi^2/t^2 - \omega^2 \right] \propto 1 - \frac{(\omega t)^2}{\pi^2 n^2}$
- Product of eigenvalues: $x / \sin x = \prod_{n=1}^{\infty} (1 - x^2/\pi^2 n^2)^{-1}$

$$\langle q_f | U(t) | q_i \rangle = \underbrace{\left(\frac{m\omega}{2\pi \mathrm{i}\hbar \sin \omega t} \right)^{1/2}}_{\text{from det...}} \underbrace{\exp\left[\frac{\mathrm{i}}{2\hbar} m\omega \left([q_i^2 + q_f^2] \cot \omega t - \frac{2 q_i q_f}{\sin \omega t} \right) \right]}_{\text{from } S[q_c]}$$

Harmonic oscillator

$$\langle q_f | \, U(t) \, | q_i \rangle = \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}[q] \exp\left(\frac{\mathrm{i}}{\hbar} \int_0^t \mathrm{d}t' \frac{1}{2} m\dot{q}^2 - \frac{1}{2} m\omega^2 q^2 \right)$$

Stationary phase ~~approximation~~ computation

- Classical paths: $q_c(t') = A \sin \omega t' + B \cos \omega t' \quad \rightarrow S[q_c]$.
- Vicinity of classical path:
  $S[q_c + r] = S[q_c] + \int_0^t \mathrm{d}t' \, r(t') \underbrace{[-\frac{m}{2}][\partial_{t'}^2 + \omega^2]} r(t')$
- Gaussian functional integral: $\det A = \prod_i a_i$ (with $A\phi_i = a_i \phi_i$)
  $\phi_i(t') = \sin(n\pi t'/t)$ and $a_i = \frac{m}{2}\left[n^2\pi^2/t^2 - \omega^2\right] \propto 1 - \frac{(\omega t)^2}{\pi^2 n^2}$
- Product of eigenvalues: $x / \sin x = \prod_{n=1}^{\infty} (1 - x^2/\pi^2 n^2)^{-1}$

$$\langle q_f | \, U(t) \, | q_i \rangle = \underbrace{\left(\frac{m\omega}{2\pi \mathrm{i} \hbar \sin \omega t} \right)^{1/2}}_{\text{from det...}} \underbrace{\exp\left[\frac{\mathrm{i}}{2\hbar} m\omega \left([q_i^2 + q_f^2] \cot \omega t - \frac{2 q_i q_f}{\sin \omega t} \right) \right]}_{\text{from } S[q_c]}$$

Energy spectrum

$$\mathrm{tr}\left(\exp\left(\frac{tH}{\mathrm{i}\hbar}\right)\right) = \int \mathrm{d}q' \left\langle q'\right| \exp\left(\frac{tH}{\mathrm{i}\hbar}\right) \left|q'\right\rangle = \sum_n \exp\left(\frac{tE_n}{\mathrm{i}\hbar}\right)$$

$$\begin{aligned}
&= \int \mathrm{d}q' \left(\frac{m\omega}{2\pi \mathrm{i}\hbar \sin\omega t}\right)^{1/2} \exp\left[\frac{-2\mathrm{i}m\omega q'^2}{\hbar}\frac{\sin^2(\omega t/2)}{\sin(\omega t)}\right] \\
&= \frac{1}{2\mathrm{i}\sin(\omega t/2)} = \left[\mathrm{e}^{\mathrm{i}\omega t/2} - \mathrm{e}^{-\mathrm{i}\omega t/2}\right]^{-1} = \frac{\mathrm{e}^{-\mathrm{i}\omega t/2}}{1-\mathrm{e}^{-\mathrm{i}\omega t}} \\
&= \sum_{n=0}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t n - \mathrm{i}\omega t\frac{1}{2}} = \sum_{n=0}^{\infty} \mathrm{e}^{\frac{t}{\mathrm{i}}\omega(n+\frac{1}{2})}
\end{aligned}$$

$$E_n = (n+\frac{1}{2})\hbar\omega$$

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Wick Rotation

Convergence: $t \mapsto -\mathrm{i}\tau$

$$\mathrm{e}^{\frac{t}{\mathrm{i}\hbar}p^2} \mapsto \mathrm{e}^{-\frac{\tau}{\hbar}p^2}$$

Statistical physics: $\frac{\mathrm{i}t}{\hbar} \mapsto \beta$

$$\mathrm{e}^{-\frac{\mathrm{i}t}{\hbar}H} \mapsto \mathrm{e}^{-\beta H}$$

$$\langle q_f| \exp\left(\frac{tH}{\mathrm{i}\hbar}\right)|q_i\rangle \to \langle q| \exp\left(-\beta H\right)|q\rangle$$

$$Z = \int \mathrm{d}x \int_{q(0)=x}^{q(\beta\hbar)=x} \mathcal{D}[q] \exp\left(-\frac{1}{\hbar}\int_0^{\beta\hbar} \mathrm{d}\tau \underbrace{\frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + V(q)}_{H}\right)$$

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Conclusion

- Operator $\rightarrow$ integral formalism via small steps and $1 = \int \mathrm{d}q \, |q\rangle \langle q|$
- Understand "$\int \mathcal{D}[q]$" as $\propto \lim_{N\to\infty} \int \mathrm{d}q_1 \cdots \mathrm{d}q_N$
- Advanced computation $\mathrm{e}^{-F[\xi]} \det \left(\frac{1}{2\pi} \frac{\delta^2 F}{\delta x(t) \delta x(t')} \right)^{-1/2}$

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