1 Generalities on phase transitions

Before entering into the subject of lattice gauge theory, let us recall some important aspects of phase transitions that will be repeatedly alluded to during this lecture. In fact, as we will see in later chapters, one of the initial motivations to develop a lattice gauge theory [1] was to address rather special phase transitions, where later, confinement [2] became one of the more important aspects.

In general phase transitions can be devided in those, where the state of the system changes in a discontinuous way, called *first order phase transitions*, and continuous ones, where the *correlation length* of the degree of freedom undergoing the phase transition diverges. We will concentrate on *continuous phase transitions*.

In general, a phase transition takes place when some intensive parameter like temperature is changed. Normally, the high temperature phase is a disordered one, with a high symmetry, and the phase transition brings the system to a lower symmetry phase. Here one speaks of a *spontaneous symmetry breaking*. Consequently, such a phenomenon is in general described in terms of an *order parameter*. It is defined such that it is zero in the disordered phase and non-zero in the ordered one. In a first order phase transition the change is discontinuous, as oppossed to a continuous phase transition.

The point at which a continuous phase transition takes place is called the *critical point*, and one speaks about *criticality* due to the fact that the correlation length diverges such that physical quantities like, e.g. the magnetic susceptibility in a magnet, diverge. This signals the fact that fluctuations are present on all length scales, i.e. there is no physical length scale characterizing the system and the system becomes *scale invariant*. This last fact makes the link between *critical phenomena* in statistical physics and elementary particle physics, where a Lorentz-invariant theory with massless particles is also scale invariant.

It turns out that three aspects characterize a critical point. The first one is the *number of components* of the order parameter. It could be a scalar (zero components), a vector (n components, where n is the dimension of the space defining the order parameter), etc. The second is the number of spatial dimensions d, where the system lives. The third one is the kind of symmetry that is broken. One big distinction is between *discrete* and *continuous symmetries*. Further distinctions are given by the symmetry groups of the high and low temperature phases.

To see this in more detail let us review some models for magnetism, where the concepts related to the number of componentes of the order parameter n and the symmetry of the phases can be seen directly. We will dedicate later some time to study the *Ising model*. We can define it on a hypercubic lattice in d dimensions as follows:

$$H = -J \sum_{\langle i,j \rangle} S_i S_j \ , \tag{1.1}$$

where $S_i = \pm 1$, and $\langle i,j \rangle$ denotes sites i and j that are nearest neighbors. We can make a transformation $S_i \to -S_i$ without changing the energy. Hence, the Hamiltonian has a global $\mathbb{Z}_2$ *symmetry*. This is a discrete symmetry. The order parameter is in this case the magnetization that will be given by

$$\langle S_i \rangle = \frac{1}{Z} \sum_{\{S_j\}} S_i \, \mathrm{e}^{K \sum_{\langle j,\ell \rangle} S_j S_\ell}, \tag{1.2}$$

where Z is the partition function and $K \equiv \beta J$, with $\beta = 1/k_B T$. It is easy to see that in the high temperature limit ($K \to 0$), all possible configurations are equally probable, and hence $\langle S_i \rangle = 0$. In the low temperature limit ($K \to \infty$) there are two configurations that maximize the exponent, namely all spin with $S_i = 1$, or all with $S_i = -1$. The system has to choose one of these two degenerate states, such that the global $\mathbb{Z}_2$ symmetry is spontaneously broken, and the order parameter is non-zero. This is actually true only if we are in the thermodynamic limit and $d \geq 2$. For $d = 1$ it can be shown that the system is allways disordered.

We can go to more general situations, where the spin is a vector. In the so-called XY-*model* we have $\boldsymbol{S}_i = (S_i^x, S_i^y)$, with say $\boldsymbol{S}_i^2 = 1$. In this case the number of components is $n = 2$. The Hamiltonian is now

$$H = -J \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \ , \tag{1.3}$$

Since applying the same O(2) rotation on each site, the energy does not change, in this case the global symmetry is O(2) or U(1). As in the previous case, the high temperature phase is disordered, while we expect the low temperature phase to be ordered. We will see below that here again the spatial dimension d plays a very important role.

Finally, we can have a *Heisenberg-model*, where $\boldsymbol{S}_i = (S_i^x, S_i^y, S_i^z)$, again with $\boldsymbol{S}_i^2 = 1$. In this case the number of components is $n = 3$. The Hamiltonian has the same form as for the XY-model but now the global symmetry is O(3). In the quantum case we would have SU(2) as the global symmetry. Until now no symmetry was playing any role in the symmetry broken phase. In this case, there is still a symmetry present in the low temperature phase. There, we expect all the spins to be pointing in one direction, i.e. we expect a *colinear phase*. But then, a rotation along that direction does not change the state, such that an O(2) or U(1) symmetry remains. The manifold of the order parameter is then the quotient $\mathrm{O}(3)/\mathrm{O}(2) \simeq S^2$. This is telling us, that the low energy modes in the broken symmetry phase are two transverse ones.

The fact that the system is scale invariant at the critical point leads to the concept of *universality* and *universality classes*. Since the system is scale invariant, microscopic details of the system become unimportant such that n, d, and the symmetries of the phases are the determining factors that characterize the transition. This is explicitely seen in the so-called *critical exponents* that acquire the same value if the systems belong to the same universality class. The critical exponents tell us e.g. how the magnetic susceptibility diverges at the transition.

Such a generality leads to the possibility of having quite powerful mathematical theorems, where even without knowing the system in detail we can make very strong statements about eventual phase transitions in them. One such theorem is the celebrated *Mermin-Wagner theorem* [3], that states that a continuous symmetry cannot be spontaneously broken in $d \leq 2$ at finite temperatures. This means that for the XY- or the Heisenberg-model discussed above, the local order parameter is zero for all temperatures $T > 0$. However, this does not mean that there is no phase transition. In fact, the XY-model shows a so-called Berezinskii-Kosterlitz-Thouless (BKT) [4, 5, 6] transition. In this case the information about a phase transition is not contained in the local order parameter but in correlation functions of it.

Let us discuss in more detail the BKT transition, in order to see the difference between the phases with vanishing local order parameter. We can rewrite the Hamiltonian (1.3) as follows:

$$H = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) = -J \sum_{\langle i,j \rangle} \left[\mathrm{e}^{i\theta_i} \mathrm{e}^{-i\theta_j} + \mathrm{h.c.} \right] , \tag{1.4}$$

Then, the information for the spin-spin correlation function is contained in the following correlator

$$\langle \mathrm{e}^{i\theta_0} \mathrm{e}^{-i\theta_n} \rangle \;=\; \frac{1}{Z} \int \prod_m \mathrm{d}\theta_m \, \mathrm{e}^{i\theta_0} \mathrm{e}^{-i\theta_n} \exp \left\{ \beta J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \right\} . \tag{1.5}$$

First we consider the high temperature limit, where a series expansion in β is possible, such that

$$\begin{aligned} &\exp \left\{ \beta J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \right\} \\ &\quad = 1 + \beta J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) + \frac{(\beta J)^2}{2!} \left[\sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \right]^2 + \cdots \end{aligned} \tag{1.6}$$

Since

$$\int_0^{2\pi} \mathrm{d}\theta = 2\pi \,, \quad \int_0^{2\pi} \mathrm{d}\theta \, \mathrm{e}^{-i\theta} = 0 \,, \tag{1.7}$$

on calculating the correlation function (1.5), we have to bring one factor that cancels $\mathrm{e}^{i\theta_0}$ together with one cancelling $\mathrm{e}^{-i\theta_n}$. But this means that the first non-zero contribution to (1.5) will appear at order $(\beta J)^{|n|}$, leading to

$$\langle \mathrm{e}^{i\theta_0} \mathrm{e}^{-i\theta_n} \rangle \;\sim\; (\beta J)^{|n|} = \mathrm{e}^{-|n| \ln(k_B T/J)} \,, \tag{1.8}$$

that is, an exponential decay of the correlations with a correlation length $\xi \sim 1/\ln(k_B T/J)$.

For the low temperature limit we expect the difference $\theta_i - \theta_j$ to be very small, so that we can expand the cosine and keep the lowest order term. In this case we have for i and j nearest neighbors,

$$\theta_i - \theta_j \simeq (\boldsymbol{\nabla}\theta_i) \cdot \boldsymbol{a}_{ij} \ , \tag{1.9}$$

where $\boldsymbol{a}_{ij}$ is the lattice vector in the direction of the bond defined by i and j, such that

$$\begin{aligned}
\langle \mathrm{e}^{i\theta_0}\mathrm{e}^{-i\theta_n}\rangle &\simeq \frac{1}{Z}\int \prod_m \mathrm{d}\theta_m \, \mathrm{e}^{i\theta_0}\mathrm{e}^{-i\theta_n} \exp\left\{\beta J \sum_i a^2 \left(\boldsymbol{\nabla}\theta_i\right)^2\right\} \\
&= \frac{1}{Z}\int \mathcal{D}\theta(x) \, \exp\left\{i\left[\theta\left(\boldsymbol{x}_0\right) - \theta\left(\boldsymbol{x}_n\right)\right]\right\} \exp\left\{\beta J \int \mathrm{d}^2x \, \left(\boldsymbol{\nabla}\theta\right)^2\right\} \\
&= \frac{1}{Z}\int \mathcal{D}\theta(x) \, \exp\left\{i \int \mathrm{d}^2x \left[\theta\left(\boldsymbol{x}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}_0\right) - \theta\left(\boldsymbol{x}\right)\delta\left(\boldsymbol{x}-\boldsymbol{x}_n\right)\right]\right\} \\
&\qquad \times \exp\left\{-\beta J \int \mathrm{d}^2x \, \theta\left(\boldsymbol{x}\right) \triangle \, \theta\left(\boldsymbol{x}\right)\right\} \ ,
\end{aligned} \tag{1.10}$$

where, apart from going to the continuum, we introduced the notation of path integrals, and the Laplacian $\triangle$, after an integration by parts. The arguments of the last exponential above can be written as follows,

$$\int \mathrm{d}^2x \, \theta\left(\boldsymbol{x}\right) \triangle \, \theta\left(\boldsymbol{x}\right) = \int \mathrm{d}^2x \, \mathrm{d}^2x' \, \theta\left(\boldsymbol{x}\right) G^{-1}\left(\boldsymbol{x},\boldsymbol{x}'\right) \, \theta\left(\boldsymbol{x}'\right) \ , \tag{1.11}$$

with the inverse *propagator*

$$G^{-1}\left(\boldsymbol{x}-\boldsymbol{x}'\right) = \delta\left(\boldsymbol{x}-\boldsymbol{x}'\right) \triangle \ . \tag{1.12}$$

In this way we see that we have an exponential of a bilinear form in θ's, so that the integral over θ can be performed. The result of such an integration is

$$\langle \mathrm{e}^{i\theta_0}\mathrm{e}^{-i\theta_n}\rangle \simeq \exp\left[\frac{k_B T}{J} G\left(\boldsymbol{x}_0 - \boldsymbol{x}_n\right)\right] \tag{1.13}$$

In order to obtain the propagator $G\left(\boldsymbol{x}-\boldsymbol{x}'\right)$, it is better to go over to Fourier space and come back:

$$G\left(\boldsymbol{x}-\boldsymbol{x}'\right) = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{\mathrm{e}^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{\boldsymbol{k}^2 + m^2} = \frac{1}{2\pi} K_0\left(m\left|\boldsymbol{x}-\boldsymbol{x}'\right|\right) \ , \tag{1.14}$$

where we introduced a *regulator* m, since without it the expression is *infrared divergent*, and $K_0(z)$ is a modified Bessel function, that in the limit $m \to 0$ behaves as

$$K_0(z) \to -\ln z \ . \tag{1.15}$$

With this result we have finally the behavior of the correlation function at low temperatures

$$\langle \mathrm{e}^{i\theta_0} \mathrm{e}^{-i\theta_n} \rangle \;\simeq\; \left(\frac{1}{|\boldsymbol{x}_0 - \boldsymbol{x}_n|} \right)^{k_B T/2\pi J} \quad . \tag{1.16}$$

In the low temperature region, the behavior of the correlation function changes from an exponential decay to a power-law. Nevertheless, it still decays to zero at large distances. This tells us that the order parameter itself is zero, since at large distances, the leading factor is

$$\langle \mathrm{e}^{i\theta_0} \mathrm{e}^{-i\theta_n} \rangle \;\sim\; \langle \mathrm{e}^{i\theta_0} \rangle \langle \mathrm{e}^{-i\theta_n} \rangle + \cdots \; , \tag{1.17}$$

with $\cdots$ corresponding to decaying functions of distance. Hence, the BKT-transition shows a different kind of phase transition than the the ordinary ones, where an order parameter emerges after spontaneous symmetry breaking

While we are not going to discuss the BKT-transition in further detail, it is interesting to recall the physics behind it. Kosterlitz and Thouless showed that there was a confinement-deconfinement transition at a temperature denoted the Kosterlitz-Thouless temperature T_{KT} [5, 6]. Above T_{KT}, topological objects (*vortices*) are free, disordering the system, and giving rise to a finite correlation length At low temperatures vortices and *antivortices* bind together due to a logarithmic interaction (with the same origin as the logarithmic behavior of the propagator, discussed above), such that they do not allow the presence of true *long-range order* but only of *quasi-long range order*, with an infinite correlation length.