

2 The Ising model, duality, and transfer matrix

In this section we still concentrate on the Ising model without gauge fields. The subjects we discuss in this chapter are, on the one hand, needed afterwards in our treatment of lattice gauge theories. In particular, the concept of duality proved on the other hand to be useful in several context in theoretical physics ranging from statistical mechanics of classical systems to string theories. Therefore, it is useful to discuss them in some detail.

2.1 Self-duality in the two-dimensional Ising model

Here we will just consider duality in the two-dimensional Ising model, as was first introduced by Kramers and Wannier [7]. A comprehensive exposition on duality can be found in the review article by Savit [8].

We start by considering the partition function of the Ising model defined in (1.1), that can be rewritten as follows:

$$Z = \sum_{\{S_j\}} e^{K \sum_{\langle j, \ell \rangle} S_j S_\ell} = \sum_{\{S_j\}} \prod_{\langle j, \ell \rangle} e^{K S_j S_\ell} = \sum_{\{S_j\}} \prod_{\langle j, \ell \rangle} \sum_{r=0}^1 C_r(K) (S_j S_\ell)^r, \quad (2.1)$$

where $C_0(K) = \cosh K$ and $C_1(K) = \sinh K$. With the simple transformations above, we see that for each bond $\langle j, \ell \rangle$, a new \mathbb{Z}_2 variable can be introduced, namely r . To recall this fact, we label it as r_μ with $\mu \equiv (i, \langle i, j \rangle)$, that is, we label it with the site i from which the bond $\langle i, j \rangle$ emanates. The partition function can now be expressed as follows

$$Z = \sum_{\{S_j\}} \sum_{\{r_\mu\}} \prod_{\langle j, \ell \rangle} C_{r_\mu}(K) \prod_i S_i^{\sum_{\langle i, j \rangle} r_\mu}, \quad (2.2)$$

where we grouped together all the products of spins on site i , such that $\sum_{\langle i, j \rangle} r_\mu$ contains all four contributions due to the bonds connected to site i . We can further perform explicitly the sum over spin configurations, leading to

$$\begin{aligned} (2.2) &= \sum_{\{r_\mu\}} \prod_{\langle j, \ell \rangle} C_{r_\mu}(K) \prod_i \sum_{S_i = \pm 1} S_i^{\sum_{\langle i, j \rangle} r_\mu} \\ &= \sum_{\{r_\mu\}} \prod_{\langle j, \ell \rangle} C_{r_\mu}(K) \prod_i 2\delta \left[\text{mod}_2 \left(\sum_{\langle i, j \rangle} r_\mu \right) \right], \end{aligned} \quad (2.3)$$

where the Kronecker delta gives 1 if the sum over bonds is even and zero if it is odd. Now we have achieved an expression for the partition function expressed only in terms of the new \mathbb{Z}_2 variable r_μ .

The presence of the Kronecker δ in (2.3) shows that many configurations have vanishing contributions. It would be more effective if we could find variables that automatically satisfy it. This can be achieved by defining a *dual lattice*, where the

vertices are set in the center of the plaquettes defined by the original one. Figure 1 shows the dual lattice corresponding to the original square lattice.

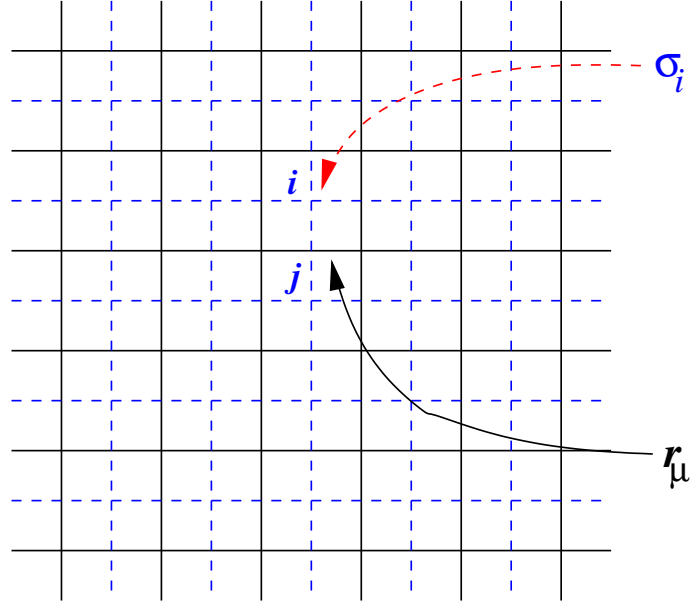


Figure 1: Dual lattice. Black full lines correspond to the original lattice, where r_μ denotes a bond. Blue dashed lines correspond to the dual lattice. The link joining the sites i and j on the dual lattice crosses the bond r_μ . New variables σ_i are defined on the sites of the dual lattice.

We define now new \mathbb{Z}_2 variables $\sigma_i = \pm 1$ on the sites of the dual lattice. For each link of the original lattice there is a pair of σ_i 's (e.g. on the sites i and j on the dual lattice in Fig. 1) that we can associate to it. We can express the variable r_μ in terms of the new variables as follows:

$$r_\mu = \frac{1}{2} (1 - \sigma_i \sigma_j) , \quad (2.4)$$

where sites i and j on the dual lattice are those whose link crosses r_μ . Then, for the sum of r_μ over the four nearest neighbors of a site i , we have

$$\sum_{\langle i,j \rangle} r_\mu = 2 - \frac{1}{2} (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_4 + \sigma_4 \sigma_1) , \quad (2.5)$$

where the indices of the σ 's were chosen as shown in Fig. 2. For the four variables $\sigma_1, \dots, \sigma_4$, there are 2^4 possible configurations. However, they can be grouped in four cases, and it turns out that all the cases lead to an even number for the sum of

r_μ over the bonds. To see this, we consider those four cases below.

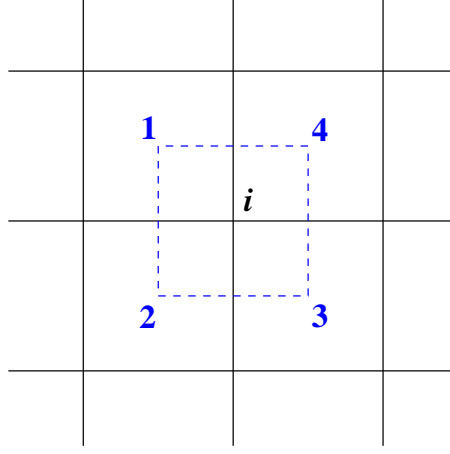


Figure 2: Four sites on the dual lattice with links crossing bonds around i .

i) $\sigma_i = 1 \quad \forall i$. An equivalent configuration is with all σ_i 's inverted. Here we have

$$\sum_{\langle i,j \rangle} r_\mu = 0. \quad (2.6)$$

ii) $\sigma_1 = \sigma_2 = \sigma_3 = -\sigma_4 = 1$. There are in total 8 equivalent configurations of this type and the result for the sum is

$$\sum_{\langle i,j \rangle} r_\mu = 2. \quad (2.7)$$

iii) $\sigma_1 = \sigma_3 = -\sigma_2 = -\sigma_4 = 1$. An equivalent configuration is obtained with all σ_i 's inverted.

$$\sum_{\langle i,j \rangle} r_\mu = 4. \quad (2.8)$$

iv) $\sigma_1 = \sigma_4 = -\sigma_2 = -\sigma_3 = 1$. Here we have in total 4 equivalent configurations, each one leading to

$$\sum_{\langle i,j \rangle} r_\mu = 2. \quad (2.9)$$

The listing above shows that with the choice of variables we made, we always satisfy the δ -function. Then, for the partition function we have.

$$Z = \frac{1}{2} 2^N \sum_{\{\sigma_i\}} \prod_{\substack{\langle j,\ell \rangle \\ \text{on dual lattice}}} C_{[(1-\sigma_i\sigma_j)/2]}(K), \quad (2.10)$$

where the factor $1/2$ is due to the fact that for each value of r_μ , there are two configurations of the σ 's, the factor 2^N comes from the factor 2 in front of the Kronecker δ in (2.3) with N the number of sites on the lattice, and the product is now over bonds in the dual lattice.

The expression of the partition function (2.10) shows that the weight for each configuration of the σ 's is given by the coefficients $C(K)$. Therefore, the next step is to see whether it is possible to bring $C(K)$ to a form that looks like a Boltzmann weight. In order to maintain a short notation, we keep here the label r_μ .

$$\begin{aligned}
C_r(K) &= \cosh K [1 + r (\tanh K - 1)] \\
&= \cosh K \exp \{ \ln [1 + r (\tanh K - 1)] \} \\
&= \cosh K \exp (r \ln \tanh K) \\
&= \cosh K \exp \left[\frac{1}{2} (1 - \sigma_i \sigma_j) \ln \tanh K \right] \\
&= (\cosh K \sinh K)^{1/2} \exp \left(-\frac{1}{2} \ln \tanh K \sigma_i \sigma_j \right) . \tag{2.11}
\end{aligned}$$

Bringing this expression into the partition function (2.10), we have

$$Z = \frac{1}{2} (\sinh 2\tilde{K})^{-N} \sum_{\{\sigma_i\}} \exp \left(\tilde{K} \sum_{\langle j, \ell \rangle} \sigma_j \sigma_\ell \right) , \tag{2.12}$$

where we used the fact that there are $2N$ bonds, and we defined the new coupling constant

$$\tilde{K} \equiv -\frac{1}{2} \ln \tanh K , \tag{2.13}$$

that is the coupling for an Ising model on the dual lattice. Hence, the Ising model

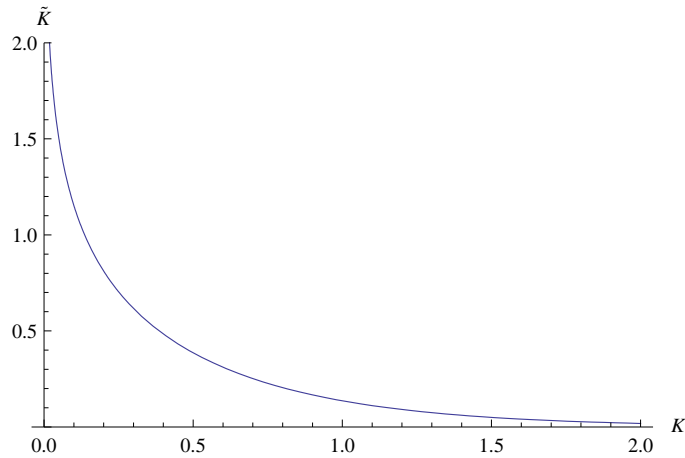


Figure 3: Relation between the original coupling K and the dual one \tilde{K} .

is *self-dual* since the duality transformation brings it into itself. Since $\tanh K < 1$, $\tilde{K} > 0$. Furthermore, when $K \rightarrow \infty$, $\tilde{K} \rightarrow 0$. The opposite happens when $K \rightarrow 0$. Therefore, the duality transformation relates the high T region (low T region) of the original model to the low T (high T) region of the dual one. This is a common feature of duality transformations, relating opposite regions of the dual models or of the same model, when it is self-dual. Such a relation is very useful, since it is in general possible to perform a high temperature expansion. The duality transformation tells then how the behavior is in the low temperature region, that is in general more difficult to extract.

Due to its self-duality, it is possible to extract for the two-dimensional Ising model the value of the critical temperature. To see this, we consider the free energy per site

$$f = -\frac{1}{N} \ln Z . \quad (2.14)$$

From the relation (2.12) between the partition functions of the original model and the dual one, we have

$$f(K) = \ln \sinh 2\tilde{K} + f(\tilde{K}) . \quad (2.15)$$

This is a rather strong constraint on the free energy. Since $\sinh 2\tilde{K}$ is an analytic function, the equation above implies that a singularity in $f(K)$ corresponds to a singularity in $f(\tilde{K})$. Furthermore, since $\tilde{K}(K)$ is a monotonous function of K , it should hold that $\tilde{K}_c = K_c$. Then, from (2.13) we have

$$e^{2K_c} = \frac{e^{2K_c} + 1}{e^{2K_c} - 1} , \quad (2.16)$$

with the solution

$$K_c = \frac{1}{2} \ln \left(1 + \sqrt{2} \right) . \quad (2.17)$$

Hence, self-duality has allowed us to calculate the exact value of the critical temperature in the two-dimensional Ising model.

A straightforward generalization of the results obtained in the isotropic case can be done for the anisotropic one, i.e. when we allow for couplings $K_x \neq K_y$ in the respective directions. This will be important for the next section, where we will see that in order to map the classical system to a quantum mechanical one, anisotropy will appear naturally. In the treatment above, we made in general no explicit use of the fact that $K_x = K_y$, since the information of the strength of the bonds was contained in the coefficients (2.11). The only place where we used isotropy was in writing eq. (2.12). However, it is easy to generalize the result there for the anisotropic case by realizing that eq. (2.11) relates the value of a bond in x -direction on the original lattice with the value of the coupling constant on the dual lattice in the y -direction. Therefore, in the anisotropic case we generalize (2.13) as follows:

$$\tilde{K}_y \equiv -\frac{1}{2} \ln \tanh K_x , \quad \tilde{K}_x \equiv -\frac{1}{2} \ln \tanh K_y . \quad (2.18)$$

Assuming again, that given K_x and K_y there is only one critical point, we can summarize both equations above into the following condition for a critical line separating the ordered from the disordered phase in an anisotropic Ising model:

$$\sinh(2K_{xc}) \sinh(2K_{yc}) = 1 . \quad (2.19)$$

The same condition is obtained in the exact solution of the anisotropic case [9], using in part the transfer matrix method, to be discussed in the next section. Figure 4 shows the phase diagram of the anisotropic Ising model.

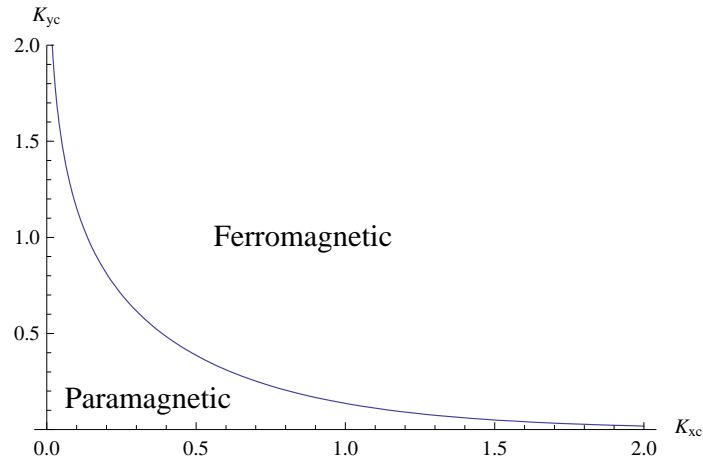


Figure 4: Phases of the anisotropic Ising model with the critical line obtained from the duality relations.

For a further more general account of duality relations, see the review article by Savit, Ref. [8].

2.2 Transfer matrix: from two dimensional classical statistics to one dimensional quantum mechanics

The transfer matrix played an important role in deriving a more transparent exact solution of the two dimensional Ising model [9] than the one originally obtained by Onsager [10], and is a well established method in statistical mechanics. Besides this, it will play an important role in models for lattice gauge theory.

We start with the partition function

$$Z = \sum_{\{S_i\}} e^{-S} , \quad (2.20)$$

with

$$S = -K \sum_{\langle i,j \rangle} S_i S_j , \quad (2.21)$$

and $S_i = \pm 1$. Since we are dealing with the model in two dimensions, it is better to introduce coordinates (p, q) for each site, where $p, q \in \mathbb{Z}$, and denote the coordinates in x - and y -direction, respectively. We generalize our model allowing for different couplings along the x - and y -directions, and having N sites in the x -direction and M -sites in the y -direction but keeping periodic boundary conditions (p.b.c) along both directions. With all these changes, the action S can be written as follows:

$$S = \sum_{q=1}^M L(q, q+1) , \quad (2.22)$$

where due to p.b.c. $M+1 \rightarrow 1$. The "Lagrangian" $L(q, q+1)$ is given by

$$L(q, q+1) = \sum_{p=1}^N (-K_x S_{p,q} S_{p+1,q} - K_y S_{p,q} S_{p,q+1}) , \quad (2.23)$$

where again, due to p.b.c. $N+1 \rightarrow 1$.

We consider first the case $K_x = 0$, that corresponds to N decoupled one-dimensional Ising models. Let us consider one of those chains, say the p -th one and at a given site q . Then, the partition function consists of a product of terms as follows,

$$Z_p = \sum_{S_{p,1}, \dots, S_{p,M}} \prod_q T_{pq}^y , \quad \text{with } T_{pq}^y = e^{K_y S_{p,q} S_{p,q+1}} . \quad (2.24)$$

Since the variables $S_{p,q}$ have two possible values, we can represent them by a two component vector (a spinor):

$$S_{p,q} = +1 \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad S_{p,q} = -1 \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad (2.25)$$

such that T_{pq}^y can be viewed as a 2×2 matrix:

$$T_{S_{p,q}, S_{p,q+1}}^y = \begin{pmatrix} e^{K_y} & e^{-K_y} \\ e^{-K_y} & e^{K_y} \end{pmatrix} . \quad (2.26)$$

Since a 2×2 matrix can be written in terms of Pauli matrices, we have

$$T_{pq}^y = e^{K_y} \mathbf{1} + e^{-K_y} \sigma_{pq}^x = e^{K_y} (\mathbf{1} + e^{-2K_y} \sigma_{pq}^x) . \quad (2.27)$$

At this point we recall that T_{pq}^y is part of a partition function, and therefore, it would be easier to interpret what we have, if we could express it as the exponential of an operator. Using the fact that $(\sigma^x)^2 = \mathbf{1}$, we have in general

$$\exp(\tilde{K}_y \sigma^x) = \cosh \tilde{K}_y + \sinh \tilde{K}_y \sigma^x = \cosh \tilde{K}_y (\mathbf{1} + \tanh \tilde{K}_y \sigma^x) . \quad (2.28)$$

Hence, setting

$$\tanh \tilde{K}_y = e^{-2K_y} , \quad (2.29)$$

we finally have

$$T_{pq}^y = \left(\sinh \tilde{K}_y \cosh \tilde{K}_y \right)^{-1/2} \exp \left(\tilde{K}_y \sigma_{pq}^x \right) . \quad (2.30)$$

This form is reminiscent of the one obtained in eq. (2.11) when we discussed duality. In fact, inverting (2.29), we have

$$e^{-2\tilde{K}_y} = \tanh K_y \quad \Rightarrow \quad \tilde{K}_y = -\frac{1}{2} \ln \tanh K_y . \quad (2.31)$$

Using (2.29) we can also rewrite the prefactor of (2.30) as

$$\sinh \tilde{K}_y \cosh \tilde{K}_y = \frac{1}{2} \sinh 2\tilde{K}_y = \frac{1}{2 \sinh 2K_y} , \quad (2.32)$$

such that

$$T_{pq}^y = (2 \sinh 2K_y)^{1/2} \exp \left(\tilde{K}_y \sigma_{pq}^x \right) . \quad (2.33)$$

Until now, we were discussing the one-dimensional Ising model. The corresponding partition function is

$$Z_p = \sum_{S_{p,1}, \dots, S_{p,M}} T_{S_{p,1}, S_{p,2}}^y T_{S_{p,2}, S_{p,3}}^y \cdots T_{S_{p,M-1}, S_{p,M}}^y T_{S_{p,M}, S_{p,1}}^y = \text{Tr} (T^y)^M , \quad (2.34)$$

where T^y is called the *transfer matrix*. Since the trace is invariant under a unitary transformation, it is more informative to look at the trace after diagonalizing T^y . In this case we have only two eigenvalues, such that

$$Z_p = \text{Tr} (T^y)^M = \lambda_1^M \left[1 + \left(\frac{\lambda_2}{\lambda_1} \right)^M \right] , \quad (2.35)$$

where we have chosen λ_1 to be the largest eigenvalue. In the thermodynamic limit, $M \rightarrow \infty$, only λ_1 remains, such that the free energy per site is given by the largest eigenvalue of the transfer matrix:

$$f = -\frac{k_B T}{M} \ln Z_p = -k_B T \ln \lambda_1 . \quad (2.36)$$

We see that as a spin-off of our discussion of the transfer matrix, we attained also an exact solution of the one-dimensional Ising model. In fact, in order to show that in this case there is no order at any temperature $T > 0$, we need to generalize slightly our action (2.21) to include a magnetic field in the form

$$S \rightarrow -K \sum_{\langle i,j \rangle} S_i S_j - h \sum_i S_i , \quad (2.37)$$

where $h = B/k_B T$, and B is the applied magnetic field. We can generalize T_{pq}^y to this case as follows:

$$T_{pq}^y \rightarrow e^{K_y S_{p,q} S_{p,q+1} + h(S_{p,q} + S_{p,q+1})/2}, \quad (2.38)$$

such that the matrix is now

$$T_{S_{p,q}, S_{p,q+1}}^y \rightarrow \begin{pmatrix} e^{K_y + h} & e^{-K_y} \\ e^{-K_y} & e^{K_y - h} \end{pmatrix}. \quad (2.39)$$

From here one can obtain the eigenvalues and hence, the free energy as a function of temperature and magnetic field. The derivative of the free energy with respect to the magnetic field gives the magnetization, and the result is that it vanishes for $B = 0$ at any temperature $T > 0$. The explicit calculation can be found in the book on statistical mechanics by K. Huang.

For the two-dimensional case, we have still to switch on K_x . Since we have already seen that an operator σ^x entered, it is also convenient to introduce a notation with bras and kets where $|S_{p,q} = \pm 1\rangle$ are eigenstates of σ^z . Then, the matrix elements of T^y can be expressed as $\langle S_{p,q} | T_{pq}^y | S_{p,q+1} \rangle$. Now we can consider two columns p and $p+1$, where we introduce an operator T^x with matrix elements

$$\langle S_{p,q} S_{p+1,q} | T_{pq}^x | S_{p,q+1} S_{p+1,q+1} \rangle = e^{K_x S_{p,q} S_{p+1,q}}. \quad (2.40)$$

Here we see that the operator T^x should be such that its matrix elements contain no information on the states at $q+1$. Furthermore, its action on the states at p and $p+1$ is

$$\begin{aligned} T_{pq}^x | S_{p,q} = 1, S_{p+1,q} = 1 \rangle &= T_{pq}^x | S_{p,q} = -1, S_{p+1,q} = -1 \rangle = e^{K_x}, \\ T_{pq}^x | S_{p,q} = 1, S_{p+1,q} = -1 \rangle &= T_{pq}^x | S_{p,q} = -1, S_{p+1,q} = 1 \rangle = e^{-K_x}. \end{aligned} \quad (2.41)$$

All the requirements above are met by

$$T_{pq}^x = \exp(K_x \sigma_{p,q}^z \sigma_{p+1,q}^z). \quad (2.42)$$

With the result above and (2.33) we arrive at the partition function for the whole system:

$$Z = (2 \sinh 2K_y)^{NM/2} \text{Tr} T^M, \quad (2.43)$$

with

$$T = \exp\left(K_x \sum_p \sigma_p^z \sigma_p^z + 1\right) \exp\left(\tilde{K}_y \sum_p \sigma_p^x\right). \quad (2.44)$$

This is now the transfer matrix for the two-dimensional anisotropic Ising model. In contrast to the one-dimensional case, we have now instead of a 2×2 matrix, a $2^N \times 2^N$

dimensional array. It is however possible to solve the problem exactly, by means of a Jordan-Wigner transformation making fermions out of spins [9]. We are not going to pursue this path but would like to explicitly show that the transfer matrix can be seen as an evolution operator in quantum mechanics, i.e. that it has the form of an exponential of a Hamiltonian. Here we follow the discussion introduced by Fradkin and Susskind [11].

We first notice that the two exponentials entering (2.44) do not commute with each other. This makes in fact the problem really quantum mechanical. However, if by some reason the arguments of both exponentials were multiplied by a small constant $\Delta\tau$, we could expand each exponential to first order in that constant and reexponentiate the result afterwards, such that the transfer matrix could be written as the exponential of an object that would correspond to a Hamiltonian. The error made acting like that would be of $\mathcal{O}(\Delta\tau^2)$, such that in the limit $\Delta\tau \rightarrow 0$, it vanishes. Going back to (2.44) this means that we have to look for the situation where both K_x and \tilde{K}_y are small. From eq. (2.29) we see that this means

$$K_x \propto \Delta\tau, \quad e^{-2K_y} \propto \Delta\tau. \quad (2.45)$$

That is, we are interested in the region $K_y \gg 1$ and $K_x \ll 1$. Although it seems to be a very special portion of parameter space, the phase diagram in Fig. 4 shows that it is still interesting, since there will be a phase transition between a disordered and an ordered phase. For definiteness, we take

$$\tilde{K}_y = \Delta\tau, \quad K_x = \lambda\Delta\tau, \quad (2.46)$$

where λ is a proportionally constant which, in principle, can take any value as long as the condition $\Delta\tau \ll 1$ is fulfilled. Performing these replacements in (2.44), we can write

$$Z \propto \text{Tr} e^{-\beta_{QM}H}, \quad (2.47)$$

where we defined $\beta_{QM} = M\Delta\tau$, the temperature in the quantum mechanical system, and

$$H = - \sum_p \sigma_p^x - \lambda \sum_p \sigma_p^z \sigma_{p+1}^z, \quad (2.48)$$

the one-dimensional Hamiltonian of the *Ising model with transverse field*, that due to the presence of two noncommuting pieces is a genuinely quantum mechanical model. The thermodynamic limit of the two-dimensional classical model corresponds to the limit $M, N \rightarrow \infty$, so that the quantum mechanical counterpart will show a phase transition only in the limit $\beta_{QM} \rightarrow 0$, i.e. only in its ground-state. This is an example of a *quantum phase transition*, where the parameter driving the phase transition is not any more the temperature but a coupling of the Hamiltonian, in this case λ . In order to obtain the critical value of λ , we recall the relation (2.19). In the limit of interest here ($K_y \gg 1$ and $K_x \ll 1$), that relation reduces to

$$2K_{xc} \frac{1}{2} e^{2K_{yc}} = 1. \quad (2.49)$$

On the other hand, in the same limit, we have from (2.29)

$$\tilde{K}_y = e^{-2K_y} , \quad (2.50)$$

such that

$$K_x = \lambda e^{-2K_y} , \quad (2.51)$$

that together with (2.49) leads to $\lambda_c = 1$. Increasing λ brings us into the ordered phase, while decreasing it leads to the disordered phase, as depicted in Fig. 4. Therefore, large values of λ correspond to the low temperature phase and small values to the high temperature one. Hence, we can identify $1/\lambda$ with temperature.

We have seen that in order to map the classical system into a quantum mechanical one, we had to search for the continuum limit in imaginary time. Yet, the critical properties of the original model could be recovered by an appropriate scaling of couplings in the spatial and temporal directions. Such a relation between quantum and classical counterparts will prove useful in understanding the phases of lattice gauge models.