3 Ising lattice gauge theory. Elitzur's theorem

In the previous chapters we have considered phase transitions in models with a global symmetry that showed a phase transition, i.e. spontaneous breaking of their global symmetry as a function of temperature or by changing a coupling constant. Here we will start the discussion of the possible phases, when the global symmetry is raised to a local one, that is, when the symmetry operation that leaves the Hamiltonian invariant is not the same on all points of the lattice but can be different from point to point. The easiest model where we can study such a situation is the Ising model, in the form introduced by Wegner [1].

3.1 Ising model with a local $\mathbb{Z}_2$ symmetry

We consider a hypercubic lattice in d-dimensional Euclidean space (or space-time if we happen to have a quantum mechanical model, as already shown for the Ising model with transverse field). The sites of the lattice are labeled by an index n. At each lattice point there are $2d$ links connecting it to the nearest neighbor sites. We label the links by (n, μ), where μ is a unit lattice vector pointing in the direction of the link. The same link can be also labeled $(n + \mu, -\mu)$. We define the model so, that now our $\mathbb{Z}_2$ variables (we call them as before spins) are sitting on the links

The next step is to define an operation that is local. In the case of the glocal $\mathbb{Z}_2$ symmetry of the Ising model we made the change $S_i \to -S_i \; \forall \; i$. Here we have spins $\sigma(n, \mu)$ and introduce a *local gauge transformation* $G(n)$ by flipping all the spins on links connected to site n. Figure 5 shows an example on a two dimensional lattice,

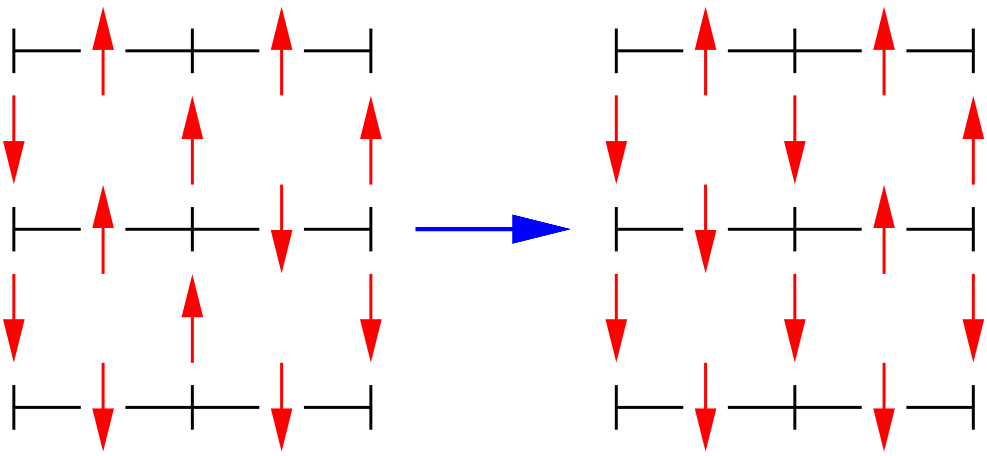

Figure 5: Gauge transformation on the central site. All the spins on the links touching the central site are flipped.

where the gauge transformation is performed at the central site of the figure.

Finally, we need an action that defines the statistical weight for the spin configurations, and is invariant under the gauge transformation. The following form fulfills those conditions:

$$S = -J \sum_{n,\mu\nu} \sigma(n,\mu)\,\sigma(n+\mu,\nu)\,\sigma(n+\mu+\nu,-\mu)\,\sigma(n+\nu,-\nu)\ . \tag{3.1}$$

Although our example in Fig. 5 shows a two dimensional lattice, the action above is defined for arbitrary dimensions. The gauge invariance of the action can be seen in Fig. 5. Since the gauge transformation flips two spins in each plaquette with a vertex at the site of the transformation, the product in each plaquette in (3.1) does change. Certainly, it would be possible to consider different forms for the action that are also gauge invariant. The form given above is the one where the interaction is as local as possible.

Due to the gauge invariance of the action, many configurations are gauge equivalent, i.e. all those configurations that can be reached from a given one through gauge transormations. In order to discern which configurations change the value of the action, we can recall the situation in the case of the ordinary Ising model. There, we looked at the configuration of each bond, where the product $S_i S_j$ was invariant under a global $\mathbb{Z}_2$ transformation. Depending on the configuration, we had there the two possible values ± 1. However, such a quantity is not gauge invariant. Instead, we can take the product of the spins in a plaquette with the two possible values ± 1:

$$\prod_{i \in p} \sigma_i = \pm 1\ , \tag{3.2}$$

where we denote with p a given plaquette. In this way we can distinguish configurations with different energies.

Once we can distinguish configurations by their energy, the question arises, whether a phase transition can take place, as in the ordinary Ising model, that is, with a spontaneous symmetry breaking. We will see in the next section, that by virtue of Elitzur's theorem, this cannot happen.

3.2 Elitzur's theorem

Spontaneous symmetry breaking is in general characterized by the development of a non-vanishing value of an order parameter. In the case of the ordinary Ising model, we have seen that the magnetization serves as a local order parameter. We will therefore, consider the local magnetization in the case of the gauge invariant model. Although we will proove its vanishing only for the Ising gauge theory, more general proofs of Elitzur's theorem [12] can be found in the book by Itzykson and Drouffe.

In order to see whether there is spontaneous symmetry breaking, we switch on a small external magnetic field h with a coupling

$$h \sum_{n,\mu} \sigma(n,\mu)\ , \tag{3.3}$$

and compute $\langle \sigma(m,\nu) \rangle$. We take the thermodynamic limit and then the limit $h \to 0$. If the expectation value of σ does not vanish, then, there is spontaneous symmetry breaking. The expectation value is

$$\langle\sigma(m,\nu)\rangle_h = \frac{1}{Z_h}\sum_{\{\sigma\}}\sigma(m,\nu)\,\exp\left[\beta J\sum_p\prod_{\ell\in p}\sigma_\ell + h\sum_{n,\mu}\sigma(n,\mu)\right] , \tag{3.4}$$

with

$$Z_h = \sum_{\{\sigma\}}\exp\left[\beta J\sum_p\prod_{\ell\in p}\sigma_\ell + h\sum_{n,\mu}\sigma(n,\mu)\right] . \tag{3.5}$$

If we perform a local gauge transformation at site m, the interaction part of the action remains unchanged due to gauge invariance. The part that couples to the magnetic field changes as follows:

$$h\sum_{n,\mu}\sigma(n,\mu) = h\sum_{n,\mu}\left[\sigma'(n,\mu) - \delta\sigma(n,\mu)\right] , \tag{3.6}$$

where σ' is the transformed spin with

$$\begin{aligned}\delta\sigma(\ell_m) &= \sigma'(\ell_m) - \sigma(\ell_m) = -2\sigma(\ell_m) ,\\ \delta\sigma(\ell_m) &= 0 \quad \text{if} \quad \ell \notin \{\ell_m\} ,\end{aligned} \tag{3.7}$$

where $\{\ell_m\}$ is the set of links emanating from m. Changing to the new variable σ' we have

$$\begin{aligned}\langle\sigma(m,\nu)\rangle_h &= -\frac{1}{Z_h}\sum_{\{\sigma'\}}\sigma'(m,\nu)\\ &\qquad\times\exp\left[\beta J\sum_p\prod_{\ell\in p}\sigma'_\ell + h\sum_{n,\mu}\sigma' - h\sum_{n,\mu}\delta\sigma\right]\\ &= \langle-\sigma(m,\nu)\,\mathrm{e}^{-h\sum_{\ell_m}\delta\sigma}\rangle_h .\end{aligned} \tag{3.8}$$

Next we look for a bound for the following quantity

$$\begin{aligned}\left|\langle\sigma(m,\nu)\rangle_h - \langle-\sigma(m,\nu)\rangle_h\right| &= \left|\langle-\sigma(m,\nu)\rangle_h\left[\mathrm{e}^{-h\sum_{\ell_m}\delta\sigma} - 1\right]\right|\\ &\leq \left|\mathrm{e}^{4dh} - 1\right|\left|\langle\sigma(m,\nu)\rangle_h\right| .\end{aligned} \tag{3.9}$$

But then, as $h \to 0$, we have

$$\langle\sigma(m,\nu)\rangle_{h=0} = \langle-\sigma(m,\nu)\rangle_{h=0} , \tag{3.10}$$

so that $\langle\sigma(m,\nu)\rangle = 0$. The difference with respect to a system with a global symmetry is that, since the system is invariant under a local transformation, such a transformation leads to a finite energy change in the presence of h. On the contrary, for a global $\mathbb{Z}_2$ symmetry, the symmetry operation involves a macroscopic number of degrees of freedom. Since local, non-gauge invariant observables do not lead to any information, we have to consider next gauge-invariant correlation functions. As we will see, such correlation functions are necessary non-local.

3.3 Gauge invariant correlation functions

Elitzur's theorem shows that it is not possible to have a spontaneous breaking of local gauge symmetries, and hence, an order-parameter cannot be used to characterize the possible phases of the system.

We have already seen the case of the XY-model, where due to Mermin-Wagner's theorem, the continuous O(2) symmetry cannot be spontaneously broken, yet we have encountered two phases. One which we called the disordered phase, where the correlation function decays exponentially, and another one below T_{KT}, the temperature at which the Kosterlitz-Thouless transition takes place, where the correlation function decays as a power law. Hence, although there is no order parameter, it is possible to distinguish different phases by looking at appropriate correlation functions. One could also distinguish them by considering the correlation length. The disordered phase has a finite correlation length while the the phase with quasi-long range order has an infinite correlation length.

In the present case, Wegner proposed to look at gauge-invariant correlation functions. As we have seen already in the construction of a gauge-invariant action for the Ising model, a gauge-invariant quantity can be obtained by considering the product of spin variables along a closed path of links,

$$\prod_{\ell \in C} \sigma(\ell) \ , \tag{3.11}$$

where the arguments of the spin variable denotes links and C is a closed contour. Two quantities related to C that are going to play an important role in the following are P, the perimeter of C, and A, the minimal area enclosed by C.

We consider now high and low temperature expansions for the Ising gauge theory. High temperature expansions are part of the established tools in statistical mechanics. We are not going to review them here, but just apply it to the case of interest without use of any formal argument. For those interested in a more detailed account on high temperature expansions, the review article by Kogut and the references therin should be appropriate.

3.3.1 High temperature expansion

Since the Boltzmann weight is an exponential, in the limit $\beta \to 0$, we could directly expand the exponential in powers of the argument. However, due to the simplicity of the variables in our case, and due to the fact that

$$\left[\prod_{\ell \in p} \sigma(\ell) \right]^2 = 1 \ , \tag{3.12}$$

we consider first the Boltzmann weight on a plaquette, where we absorb the coupling constant J in the definition of temperature, and make a simple exact transformation:

$$
\begin{aligned}
e^{\beta\sigma\sigma\sigma\sigma} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\beta \prod_{\ell\in p} \sigma(\ell)\right]^n \\
&= \cosh\beta + \sinh\beta \prod_{\ell\in p} \sigma(\ell) \\
&= \left[1 + \tanh\beta \prod_{\ell\in p} \sigma(\ell)\right] \cosh\beta \; .
\end{aligned}
\tag{3.13}
$$

Once we transformed the weight for a plaquette, we can write a thermodynamic average as

$$
\left\langle \prod_{\ell\in C} \sigma(\ell) \right\rangle = \frac{\sum_{\{\sigma\}} \prod_{\ell\in C} \sigma(\ell) \prod_p \left[1 + \tanh\beta \prod_{\ell\in p} \sigma(\ell)\right]}{\sum_{\{\sigma\}} \prod_p \left[1 + \tanh\beta \prod_{\ell\in p} \sigma(\ell)\right]} \; . \tag{3.14}
$$

At high temperatures, we can expand the expression above in powers of $\tanh\beta$. In lowest order we have

$$
\begin{aligned}
\sum_{\{\sigma\}} \prod_{\ell\in C} \sigma(\ell) &= \sum_{\{\sigma\in C\}} \prod_{\ell\in C} \sigma(\ell) \sum_{\{\sigma\notin C\}} 1 \\
&= \prod_{\ell\in C} \sum_{\sigma(\ell)=\pm 1} \sigma(\ell) \sum_{\{\sigma\notin C\}} 1 = 0 \; .
\end{aligned}
\tag{3.15}
$$

Since

$$
\sum_{\sigma} \sigma^2 = 2 \; , \tag{3.16}
$$

in order to obtain a non-zero result, we have to be able to have squared all the σ's on C. We can reach that by considering as many powers of $\tanh\beta$ as plaquettes needed to follow the contour C. However, although the σ variables on C get squared, there remain unpaired spins in the inside of the contour. In order to obtain a non-vanishing result, we have to pair them also. Therefore, the first non-zero contribution in the numerator of (3.14) is obtained when all links contained in C and in its interior appear twice, i.e. when we have a contribution containing all plaquettes in the minimal surface bounded by C:

$$
\begin{aligned}
\left\langle \prod_{\ell\in C} \sigma(\ell) \right\rangle &\propto (\tanh\beta)^{N_{p_C}} + \text{higher order} \\
&= e^{N_{p_C} \ln\tanh\beta} + \cdots \; ,
\end{aligned}
\tag{3.17}
$$

where N_{p_C} is the number of plaquettes inside C. There are going to be also higher order contributions but all of them should appear in such a way that all the σ's on the links in the interior of C are paired. In general we can write

$$\left\langle \prod_{\ell \in C} \sigma(\ell) \right\rangle = \mathrm{e}^{-f(\beta)A} , \tag{3.18}$$

where

$$f(\beta) = -\ln \tanh \beta + \cdots , \tag{3.19}$$

and A is the minimal area subtended by C. This is the celebrated *area law* that holds for finite but high temperatures. In the case of the ordinary Ising model, it can be proved that the high temperature expansion has a finite radius of convergence, and the same applies here. While we state this without demonstration, more information can be gathered in the review by Kogut.

3.3.2 Low temperature expansion

Since we will see later that $d = 2$ is a special case, we start the discussion by considering $d > 2$.

In the case of the ordinary Ising model, we know that the low temperature phase is the one with all spins pointing in the same direction. In the present case, we have the possibility of performing local gauge transformations, such that many other (in principle infinite many in the thermodynamic limit) configurations can be generated from that one, without changing the energy of the system. It is therefore convenient to reduce this redundancy by taking representatives of classes defined by those configurations that are gauge equivalent, that is the configurations that can be connected by a gauge transformation. Since we are computing gauge-invariant expectation values, by just considering one representative of a given class, the result does not depend on the particular representative. On summing over all possible configurations of spins, a common factor emerges for each class, that is taken care of by the normalization. Hence, after having chosen a representative for a gauge-invariant configuration, we can speak of spin-flips, as long as such changes do not correspond to a gauge transformation.

In the limit $\beta \to \infty$, it is clear that we maximize the Boltzmann weight by taking all spins equal, say $\sigma = 1$, and hence, we take this state as a representative of the state at $T = 0$. Then, the expansion at low temperatures can proceed by considering the number of flipped spins. For the expectation value we have as allways

$$\left\langle \prod_{\ell \in C} \sigma(\ell) \right\rangle = \frac{1}{Z} \sum_{\{\sigma\}} \prod_{\ell \in C} \sigma(\ell) \exp\left[\sum_{p} \prod_{\ell \in p} \sigma(\ell) \right] . \tag{3.20}$$

When all spins are "up", we have

$$\left\langle \prod_{\ell \in C} \sigma(\ell) \right\rangle = 1 . \tag{3.21}$$

Let us now flip one spin. Since each link belongs to $2(d-1)$ plaquettes, this gives the number of unsatisfied plaquettes, that we call *frustrated.* We call the weight of the configuration with all spins up

$$W_\uparrow = \mathrm{e}^{\beta N_p} \ , \tag{3.22}$$

where N_p is the number of plaquettes in the system. Then, by considering the expansion in the number of flipped spins we have

$$\frac{Z}{W_\uparrow} = 1 + N_\ell \, \mathrm{e}^{-4(d-1)\beta} + \cdots \ , \tag{3.23}$$

where N_ℓ is the number of links in the system, that appears since Z contains a sum over all possible configurations.

For the observable $\prod_{\ell \in C} \sigma(\ell)$ we will have a contribution -1, whenever the flipped spin is on C. If L is the number of links in C, we have

$$\left\langle \prod_{\ell \in C} \sigma(\ell) \right\rangle = \frac{1 + (N_\ell - 2L)\, \mathrm{e}^{-4(d-1)\beta} + \cdots}{1 + N_\ell \, \mathrm{e}^{-4(d-1)\beta} + \cdots} \ . \tag{3.24}$$

For n flipped spins, considering them as totally independent, we would have for the numerator

$$\frac{1}{n!} (N_\ell - 2L)^n \, \mathrm{e}^{-4n(d-1)\beta} \ , \tag{3.25}$$

where, since we consider the events as independent, the total weight is just the product of the weights. Furthermore, since the order of the flipped spins does not matter, we have a factor $1/n!$. On the other hand, the partition function itself will have contributions due to flipped spins that go as

$$\frac{1}{n!} N_\ell^n \, \mathrm{e}^{-4n(d-1)\beta} \ . \tag{3.26}$$

Summing over n, we have

$$\begin{aligned} \left\langle \prod_{\ell \in C} \sigma(\ell) \right\rangle &= \frac{\sum_n \frac{1}{n!} (N_\ell - 2L)^n \, \mathrm{e}^{-4n(d-1)\beta}}{\sum_n \frac{1}{n!} N_\ell^n \, \mathrm{e}^{-4n(d-1)\beta}} \\ &= \frac{\exp\left[(N_\ell - 2L)\, \mathrm{e}^{-4(d-1)\beta}\right]}{\exp N_\ell \, \mathrm{e}^{-4(d-1)\beta}} = \exp\left[-2\mathrm{e}^{-4(d-1)\beta} L\right] \ . \end{aligned} \tag{3.27}$$

We have obtained in this way the so-called *perimeter law.* A more accurate determination would have led to

$$\langle \prod_{\ell \in C} \sigma(\ell) \rangle = \mathrm{e}^{-h(\beta) L} \ , \tag{3.28}$$

with $h(\beta) = 2 \exp\left[-4(d-1)\beta\right]$ in leading order. Such an accurate determination was performed in Ref. [1] and shows that beyond the approximations made here, the perimeter law holds.

We discuss now the special features of the $d = 2$ case. As in the preceeding discussion, we flip one spin on a given link, and as we mentioned already, there are going to be $2(d-1) = 2$ frustrated plaquettes. However, in this case it is possible to invert a line of spins and there are still only two plaquettes that are frustrated. Figure 6 displays an example showing that the frustrated plaquettes are at the end of the line of down spins. Hence, there is a special degeneracy in this case, where

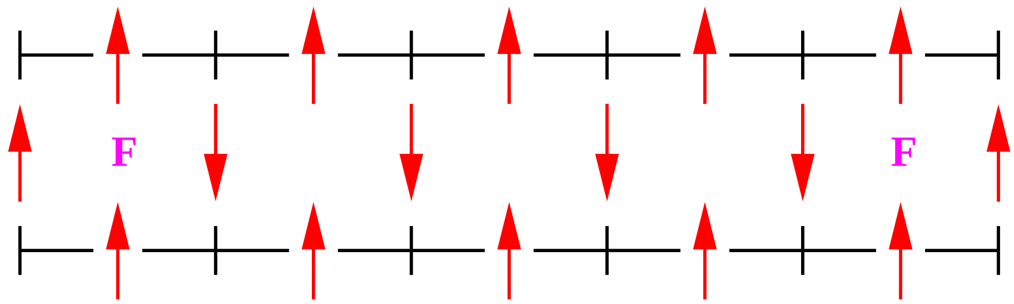

Figure 6: Example of a configuration with two frustrated plaquettes denoted by **F** at the ends of the line of overturned spins.

many configurations, independently of the number of overturned spins leads to the same action.

We can learn more about this special case by looking at the gauge-invariant correlation function we defined before. We consider a system in the thermodynamic limit and focus on the case where the line of overturned spins has one end at infinity. This is in fact one of the possible degenerate configurations. There are two possibilities for the contour C of our correlation function. Either it contains the end of the overturned spins in its inside, or not. If not, at most an even number of spins on C are overturned since at most, the line of overturned spins enters and leaves the space enclosed by C. All these contributions will lead to the same value as in the case where no spin is inverted. On the other hand, if the end of the string of overturned spins is inside C, there is an odd number of inverted spins on C, leading to a negative contribution for the correlation function. We can sum over all possible configurations in a way similar to the low temperature expansion for $d > 2$. The result for the correlation function is

$$\left\langle \prod_{\ell \in C} \sigma(\ell) \right\rangle = \frac{1 + (N - 2N_C)\,\mathrm{e}^{-4\beta} + \cdots}{1 + N\,\mathrm{e}^{-4\beta} + \cdots}\,, \tag{3.29}$$

where N_C is the number of links inside C, i.e. the number of possibilitites for the string to end inside C. As before, we assume for the case with n strings that they are not correlated such that their contribution is

$$\begin{aligned} \left\langle \prod_{\ell \in C} \sigma(\ell) \right\rangle &\simeq \frac{\exp\left[(N - 2N_C)\mathrm{e}^{-4\beta}\right]}{\exp\left(N\mathrm{e}^{-4\beta}\right)} \\ &= \exp\left(-2\,\mathrm{e}^{-4\beta} N_C\right) \simeq \exp\left(-4\,\mathrm{e}^{-4\beta} A\right)\,, \end{aligned} \tag{3.30}$$

where we set $A \simeq N_C/2$. Hence, we obtain in this case the same as in the high temperature expansion, or equivalently, we do not see any signal of a phase transition.

We can gain further insight into the result for the two dimensional case by viewing it as living in space-time of a one-dimensional model, where one direction is space-like with links denoted by $\hat{x}$ and the other time-like, with links labeled by $\hat{\tau}$. Since we are interested in gauge-invariant correlations, the results should be independent of a particular choice of gauge. We can then choose, i.e. fix the gauge to our convenience. In this case we choose

$$\sigma(n, \hat{\tau}) = 1 \tag{3.31}$$

for each temporal link. In fact, given any configuration we can perform a gauge transformation such that the condition above is fulfulled. This is called the *temporal gauge*. The action in this gauge is

$$S = -J \sum_{\substack{n \\ \mu=\hat{x},\hat{\tau}}} \sigma(n, \hat{x})\, \sigma(n + \hat{\tau}, \hat{x}) \; . \tag{3.32}$$

In this gauge the couplings in the horizontal direction do not appear any more, such that a one-dimensional Ising model in the $\hat{\tau}$ direction results. Therefore we have the following remarkable relation:

$$\text{2D Ising gauge theory} = \text{1D Ising model}$$

As we discussed before, the one-dimensional Ising model is disordered at all finite temperatures. This confirms our discussion, where we found an area law also at low temperatures for the gauge invariant correlation function.

We can also evaluate the correlation function on a simple rectangular contour with a width R along the spatial and a height T along the temporal direction. Then, for the product of spins along the contour we have

$$\begin{aligned} \prod_{\ell \in C} \sigma(\ell) &= \sigma(0, 0, \hat{x})\, \sigma(0, 1, \hat{x}) \cdots \sigma(0, R, \hat{x}) \\ &\quad \vdots \\ &\quad \times \sigma(T, 0, \hat{x})\, \sigma(T, 1, \hat{x}) \cdots \sigma(T, R, \hat{x}) \; , \end{aligned} \tag{3.33}$$

since all the vertical links are one. Since there is no interaction on the horizontal direction, we have the spin-spin correlation on the τ-direction R times. This means

$$\langle \prod_{\ell \in C} \sigma(\ell) \rangle = \langle \sigma(T, 0, \hat{x})\, \sigma(0, 0, \hat{x}) \rangle^R \sim \mathrm{e}^{-TR/\xi} = -\mathrm{A}/\xi \; . \tag{3.34}$$

3.4 Quantum Hamiltonian and duality transformation of the three-dimensional Ising gauge theory

In order to deepen our understanding of the phases that can occur in the Ising gauge theory, we consider first the mapping to a τ-continuum quantum mechanical Hamiltonian, as we did in Sec. 2.2 for the anisotropic two-dimensional Ising model. As in that case, we start by generalizing the action by considering anisotropic couplings:

$$S = -J_\tau \sum_{p_\tau} \prod_{\ell \in p_\tau} \sigma_\ell - J_s \sum_{p_s} \prod_{\ell \in p_s} \sigma_\ell \ , \qquad (3.35)$$

where we denote with p_τ plaquettes containing links in the temporal direction and with p_s plaquettes containing links only in the spatial direction. It is convenient to choose the temporal gauge, that was introduced at the end of the previous Sec. 3.3.2 linking the 2D Ising gauge theory and the 1D Ising model, and expressed in eq. (3.31). Although we fixed the gauge, there is still a large set of local gauge operations under which the action is invariant, namely all τ-independent gauge transformations. In this case, given a site n on which the gauge transformation should be performed, all other sites with the same spatial coordinates along the time direction will be affected. But this means that the spins are going to be flipped twice on the temporal link, such that the condition (3.31) is respected. Then, the temporal part of the action above simplifies as follows:

$$-J_\tau \sum_{p_\tau} \sigma(n,\hat{x})\, \sigma(n+\tau,\hat{x}) \quad \rightarrow \quad \frac{J_\tau}{2} \sum_{p_\tau} \left[\sigma(n+\tau,\hat{x}) - \sigma(n,\hat{x})\right]^2 \ , \qquad (3.36)$$

where passing to the last expression, we just performed a shift of the energy. After these transformations, the action reads now

$$S = \frac{J_\tau}{2} \sum_{p_\tau} \left[\sigma(n+\tau,\hat{x}) - \sigma(n,\hat{x})\right]^2 - J_s \sum_{p_s} \prod_{\ell \in p_s} \sigma_\ell \ . \qquad (3.37)$$

We can now construct the transfer matrix for the model in the same way as we did in Sec. 2.2. Focusing first on the temporal part of the action, i.e. assuming $J_s = 0$, and introducing a spinor notation for the σ-variables, we have now a transfer matrix of the form

$$T^\tau = \begin{pmatrix} 1 & \mathrm{e}^{-2\beta J_\tau} \\ \mathrm{e}^{-2\beta J_\tau} & 1 \end{pmatrix} = \mathbf{1} + \mathrm{e}^{-2\beta J_\tau} \sigma^x = \left(\cosh \tilde{K}_\tau\right)^{-1} \exp\left(\tilde{K}_\tau\, \sigma^x_{n,\hat{x}}\right) \ , \qquad (3.38)$$

with

$$\tanh \tilde{K}_\tau = \mathrm{e}^{-2\beta J_\tau} \ . \qquad (3.39)$$

With the same reasoning we had to arrive to (2.42), we just replace the variables by operators in the term proportional to J_s.

The last step to reach a Hamiltonian is to consider the limit where βJ_s and $\tilde{K}_\tau = \mathrm{e}^{-2\beta J_\tau}$ are small, i.e. $\beta J_s \ll 1$ and $\beta J_\tau \gg 1$. In the same way as for the two-dimensional Ising model in Sec. 2.2, we have finally a Hamiltonian of the form

$$H = -\sum_{n,\hat{x}} \sigma^x(n,\hat{x}) - \lambda \sum_{p_s} \prod_{\ell \in p_s} \sigma^z_\ell \ , \qquad (3.40)$$

where now we have a pure spatial lattice two-dimensional lattice.

Once we have reached a quantum mechanical description of the system, we can also look at the local symmetries of the theory in an operator formulation. An operator that flips all the links emanating from a given site n is given by

$$G(n) = \prod_{\pm\hat{x}} \sigma^x(n, \hat{x}) \; . \tag{3.41}$$

This operator commutes with the first term in (3.40), so that we need only to consider its action on the σ^z operator. There we have

$$G^{-1}(n)\, \sigma^z(n, \hat{x})\, G(n) = -\sigma^z(n, \hat{x}) \; . \tag{3.42}$$

Then it follows that the Hamiltonian is invariant under such operation

$$G^{-1}(n)\, H\, G(n) = H \; , \tag{3.43}$$

that is, the Hamiltonian is gauge invariant. Elitzur's theorem implies that the Hilber space is invariant to local gauge operations:

$$G(n) \mid \psi\rangle = \mid \psi\rangle \; . \tag{3.44}$$

As a consequence, we can see explicitely that spontaneous symmetry breaking is not possible:

$$\begin{aligned} \langle\psi \mid \sigma^z(n, \hat{x}) \mid \psi\rangle &= \langle\psi \mid G(n)\, G^{-1}(n)\, \sigma^z(n, \hat{x})\, G(n)\, G^{-1}(n) \mid \psi\rangle \\ &= -\langle\psi \mid \sigma^z(n, \hat{x}) \mid \psi\rangle \; , \end{aligned} \tag{3.45}$$

such that $\langle\psi \mid \sigma^z(n, \hat{x}) \mid \psi\rangle = 0$.

As a consequence of the invariance (3.44), the operator $G(n)$ can be viewed as the identity, such that

$$\sigma^x(n, \hat{y})\, \sigma^x(n, -\hat{y})\, \sigma^x(n, \hat{x})\, \sigma^x(n, -\hat{x}) = 1 \; . \tag{3.46}$$

We can use this property in order to reduce the number of operators that appear explicitely, since from the relation above, we can write

$$\sigma^x(n, \hat{y}) = \sigma^x(n, \hat{x})\, \sigma^x(n, -\hat{x})\, \sigma^x(n, -\hat{y}) \; . \tag{3.47}$$

The last operator can be again written in terms of the operators on the links arriving at $n - \hat{y}$, such that

$$\begin{aligned} \sigma^x(n, \hat{y}) &= \sigma^x(n, \hat{x})\, \sigma^x(n, -\hat{x})\, \sigma^x(n - \hat{y}, \hat{x}) \\ &\quad \times \sigma^x(n - \hat{y}, -\hat{x})\, \sigma^x(n - \hat{y}, -\hat{y}) \; . \end{aligned} \tag{3.48}$$

This means, that we can eliminate operators σ^x on links pointing in the y-direction by applying recursively (3.47). After having eliminated such operators from the Hamiltonian, we have to realize that the operators σ^z on those links, i.e. pointing also in the y-direction commute with H, and hence, the expectation values of these operators are conserved quantities. Therefore, without altering the physics, we can choose them to be a constant

$$\sigma^z(n,\hat{y}) = 1 \; . \tag{3.49}$$

As a result of the manipulations above, $\sigma^x(n,\hat{x})$ and $\sigma^z(n,\hat{x})$ are the only operators appearing explicitely in H.

We can now define a duality transformation for the three-dimensional Ising gauge theory. As already performed in the case of the two-dimensional Ising model, we define adual lattice with sites at the center of the original lattice. On such sites we define further a "dual spin-flip" operator by

$$\mu^x(n^*) = \prod_{\ell \in p} \sigma^z(\ell) \; , \tag{3.50}$$

where p is the plaquette associated with n^*. A corresponding operator μ^z can also defined as follows

$$\mu^z(n^*) = \prod_{n' \geq 0} \sigma^x(n - n'\hat{y}, \hat{x}) \; . \tag{3.51}$$

We have to see now, whether such definitions are consistent with the Pauli spin-algebra. First we can easily see that

$$\left[\mu^x(n^*)\right]^2 = \left[\mu^z(n^*)\right]^2 = 1 \; . \tag{3.52}$$

Furthermore, for the anticommutator on the same site we have

$$\{\mu^x(n^*), \mu^z(n^*)\} = 0 \; , \tag{3.53}$$

since, going back to the original operators, we see that there is only one link in common for the two dual opertors, such that on that link $\{\sigma^x(n,\hat{x}), \sigma^z(n,\hat{x})\} = 0$, where n is a site on the original lattice (say at the lower left corner of the corresponding plaquette). Finally, we have to see whether these operators commute on different sites. In fact, since a problem can only arise if both sites are on the same column along the y-direction, and μ^x involves the product of two operators σ^z on links along the x-direction, the commutator will involve two commutation operations of the original operators. Hence,

$$[\mu^x(n^*), \mu^z(m^*)] = 0 \; , \quad \text{for} \quad n^* \neq m^* \; . \tag{3.54}$$

Once we have seen that the mapping reproduces faithfully the spin algebra, we consider how to express the Hamiltonian with the new operators. From the definition (3.51) it clearly follows that

$$\mu^z(n^*)\mu^z(n^* - \hat{y}) = \sigma^x(n,\hat{x}) \; . \tag{3.55}$$

On the other hand,

$$\begin{aligned}\mu^z(n^*)\mu^z(n^*-\hat{x}) &= \sigma(n,\hat{x})\sigma(n-\hat{y},\hat{x})\sigma(n-2\hat{y},\hat{x})\cdots \\ &\quad \times\sigma(n-\hat{x},\hat{x})\sigma(n-\hat{x}-\hat{y},\hat{x})\sigma(n-\hat{x}-2\hat{y},\hat{x})\cdots \\ &= \sigma(n,\hat{x})\sigma(n-\hat{y},\hat{x})\sigma(n-2\hat{y},\hat{x})\cdots \\ &\quad \times\sigma(n,-\hat{x})\sigma(n-\hat{y},-\hat{x})\sigma(n-2\hat{y},-\hat{x})\cdots \\ &= \sigma^x(n,\hat{y}) \; , \end{aligned} \tag{3.56}$$

where the last equality comes from (3.48), when the recursion relation is used to eliminate $\sigma^x(n-n'\hat{y},\hat{y}) \;\forall\; n>0$. Putting together all the pieces, we finally have

$$\begin{aligned} H &= -\sum_{n^*,\ell}\mu^z(n^*)\mu^z(n^*+\ell) - \lambda\sum_{n^*}\mu^x(n^*) \\ &= \lambda\left[-\sum_{n^*}\mu^x(n^*) - \frac{1}{\lambda}\sum_{n^*,\ell}\mu^z(n^*)\mu^z(n^*+\ell)\right] \; , \end{aligned} \tag{3.57}$$

where the expression in brackets is the Hamiltonian version of the three-dimensional anisotropic Ising model with coupling λ^{-1}. Viewing λ as inverse temperature, as we did at the end of Sec.2.2, we have a mapping between the high (low) temperature properties of the gauge system and the low (high) temperature behavior of the conventional Ising model.

It is well known that the three-dimensional Ising model undergoes a continuous phase transition at a critical temperature T_c and develops long-range order. It is then possible to characterize the phases of the Ising gauge model with help of the order parameter of the ordinary Ising model. The order parameter of the Ising model and the corresponding quantity for the Ising gauge theory are related as follows

$$\langle 0 \mid \mu^z(n^*) \mid 0\rangle = \langle 0 \mid \prod_{n'\geq 0}\sigma^x(n-n'\hat{y},\hat{x}) \mid 0\rangle \; , \tag{3.58}$$

such that we can now label the phases of the gauge theory by means of a *nonlocal disorder parameter* that becomes non-zero in the high temperature phase since the order parameter of the ordinary Ising model acquires a non-zero value in that region of the gauge theory. This can be summarized as

$$\begin{aligned} \langle 0 \mid \prod_{n'\geq 0}\sigma^x(n-n'\hat{y},\hat{x}) \mid 0\rangle &= 0 \quad (\lambda \text{ large}) \; , \\ \langle 0 \mid \prod_{n'\geq 0}\sigma^x(n-n'\hat{y},\hat{x}) \mid 0\rangle &\neq 0 \quad (\lambda \text{ small}) \; . \end{aligned} \tag{3.59}$$

In order to understand this result, we can think of the operator

$$\prod_{n'\geq 0}\sigma^x(n-n'\hat{y},\hat{x}) \tag{3.60}$$

as a *kink operator*. To see this we can think on the action of this operator on a fully ordered state with, say, all spins up. The operator above will flip all the spins starting at the link $(n, \hat{x})$ down to the lowest link in the system. Its expectation value for large λ is zero, meaning that the state is free of kinks. Above the critical point, its expectation value is non-zero, a fact that can be understood as a kink condensate. Hence, such a state is characterized by topologically non-trivial objects, the kinks.

In spite of its close connection to the local order parameter of the ordinary Ising model, the fact that the disorder parameter above acquires a non-vanishing value does not contradict Elitzur's theorem, since it is non-local. Furthermore, considering the operator for local gauge transformations (3.41), we see that the expectation value obtained is gauge invariant since both operators commute. Finally, once we understand that the high temperature phase is a kink condensate, we can go back to the gauge invariant correlation function (3.11), and consider a purely spatial contour C. The fact that we have kinks, means that also frustrated plaquettes are present. Then, repeating the reasoning we had for the two-dimensional Ising gauge theory, we can arrive at the already discussed area law.