4 Abelian lattice gauge theory

As a next step we consider a generalization of the discussion in Ch. 3 from a discrete to a continuous symmetry. The simplest case is given by an invariance of the theory under a U(1) local symmetry, i.e. by an abelian lattice gauge theory, that can be viewed as a lattice version of electrodynamics.

4.1 U(1) lattice gauge theory

Since in our discussion of the $\mathbb{Z}_2$ lattice gauge theory we upgraded the global $\mathbb{Z}_2$ symmetry of the Ising model to a local one, we proceed in a similar way here starting with the XY-model. The spin variable is confined to a plane with two components, that, as already discussed in Ch. 1, are given by

$$\boldsymbol{S}(n) = \begin{bmatrix} \cos\theta(n) \\ \sin\theta(n) \end{bmatrix}, \tag{4.1}$$

where the angle θ is measured with respect to a given, globally defined axis. The ordinary XY-model has an action, with the notation introduced in Ch. 3 given by

$$S = -J\sum_{n,\mu} \boldsymbol{S}(n)\cdot\boldsymbol{S}(n+\mu) = -J\sum_{n,\mu}\cos\left[\theta(n)-\theta(n+\mu)\right] \ . \tag{4.2}$$

It is therefore convenient to introduce a difference operator

$$\Delta_\mu\,\theta(n) = \theta(n+\mu)-\theta(n)\ , \tag{4.3}$$

so that the action is now

$$S = -J\sum_{n,\mu}\cos\left[\Delta_\mu\,\theta(n)\right] \ . \tag{4.4}$$

As already discussed in Ch. 1, this action has an U(1) global symmetry.

The general scheme to elevate the global symmetry to a local one, follows the same steps as in the previous chapter. We place planar spins on the links (n,μ) of a lattice that we denote $\theta_\mu(n)$, Since the same link can also be labeled $(n+\mu,-\mu)$, we have to give the relationship between $\theta_\mu(n)$ and $\theta_{-\mu}(n+\mu)$. Our convention is

$$\theta_{-\mu}(n+\mu) = -\theta_\mu(n)\ . \tag{4.5}$$

With such a convention we can interprete $\theta_\mu(n)$ as giving the angle of a reference axis at site $n+\mu$ with respect to the reference axis at site n. On changing to the point of view that the angular variable on that link gives the angle of the reference axis on site n with respect to the reference axis on $n+\mu$, i.e. considering $\theta_{-\mu}(n+\mu)$, a minus sign must be introduced. We note also that with such a convention, the variables appear as in Fig. 7, when considering a directed path around a plaquette.

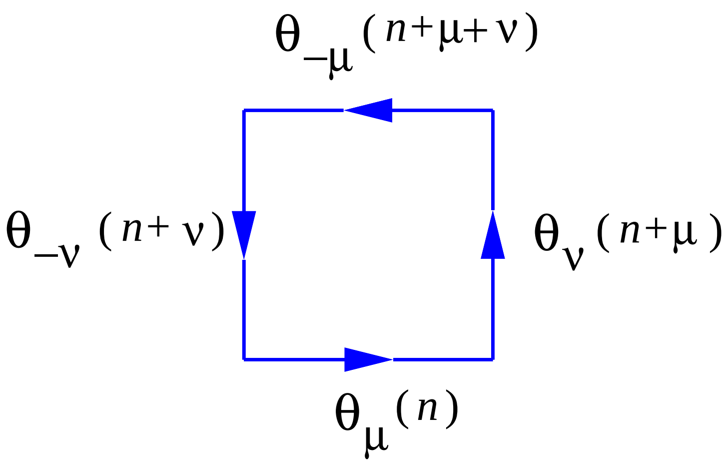

Figure 7: Directed plaquette with U(1) variables.

In analogy to electrodynamics, we can introduce a curl, that in this case should be defined as a discrete difference.

$$\begin{aligned}
\theta_{\mu\nu}(n) &= \Delta_\mu \theta_\nu(n) - \Delta_\nu \theta_\mu(n) \\
&= \theta_\nu(n+\mu) - \theta_\nu(n) - \theta_\mu(n+\nu) + \theta_\mu(n) \\
&= \theta_\mu(n) + \theta_\nu(n+\mu) + \theta_{-\mu}(n+\mu+\nu) + \theta_{-\nu}(n+\nu) \ , \qquad (4.6)
\end{aligned}$$

where we used on passing to the last line, the convention (4.5). With that convention, the curl introduced above leads to the sum over the angular variables around a directed plaquette. As in electrodynamics, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$, the discrete curl is also invariant upon a gauge transformation on a site n, defined in analogy to the Ising gauge theory, by a rotation of an angle $\chi(n)$ of all the spins on the links emanating from that site. Such a transformation leads to

$$\theta_\mu(n) \to \theta_\mu(n) - \chi(n) \qquad (4.7)$$

on the link (n, μ), while

$$\theta_{-\nu}(n+\nu) \to \theta_{-\nu}(n+\nu) + \chi(n) \qquad (4.8)$$

on the link (n, ν). The similarity with electrodynamics becomes more evident by performing a gauge transformation on site n by $\chi(n)$ and at site $n+\mu$ by $\chi(n+\mu)$. Then we have

$$\theta_\mu(n) \to \theta_\mu(n) - \chi(n) + \chi(n+\mu) = \theta_\mu(n) + \Delta_\mu \chi(n) \ , \qquad (4.9)$$

while the curl remains invariant.

Following the close correspondence between $F_{\mu\nu}$ and $\theta_{\mu\nu}$, we define the action as follows:

$$S = J \sum_{n,\mu\nu} [1 - \cos\theta_{\mu\nu}(n)] \ , \tag{4.10}$$

that is gauge invariant and periodic in the variable $\theta_{\mu\nu}$. In fact, considering the low temperature limit we expect $\theta_\mu(n)$ to be smoothly varying, so that $\theta_{\mu\nu}(n)$ is small, and the cosine can be expanded, such that

$$1 - \cos\theta_{\mu\nu}(n) \simeq \frac{1}{2}\theta_{\mu\nu}^2 \ , \tag{4.11}$$

and since the fields vary smoothly, the sum over links can be converted into an integral such that the action becomes

$$S \rightarrow \frac{J}{2} \int \frac{\mathrm{d}^d x}{a^d} \theta_{\mu\nu}^2 \ . \tag{4.12}$$

We can further identify

$$\theta_{\mu\nu} = a^2 g F_{\mu\nu} \ , \quad J = \frac{1}{2g^2} \ , \tag{4.13}$$

such that

$$S = \frac{1}{4a^{d-4}} \int \mathrm{d}^d x \, F_{\mu\nu} F_{\mu\nu} \ , \tag{4.14}$$

becomes the Euclidean action of electrodynamics. Taking into account (4.6), the identities above lead to the relation with the electrodynamic potential:

$$\theta_\mu(n) = a g A_\mu(x) \ . \tag{4.15}$$

The identifications made above with electrodynamics will be usefull later, when the coupling of matter to the gauge fields is discussed.

4.2 Gauge-invariant correlation functions and phase diagrams

In order to study correlation functions and the phase diagram, we consider the partition function of the theory, given by

$$Z = \int_0^{2\pi} \prod_{n,\mu} \mathrm{d}\theta_\mu(n) \, \exp\left\{ -\frac{1}{2g^2} \sum_{n,\mu\nu} [1 - \cos(\Delta_\mu\theta_\nu - \Delta_\nu\theta_\mu)] \right\} \ . \tag{4.16}$$

Although we do not demonstrate it here, Elitzur's theorem is also valid in this case (in fact he did it explicitely for the U(1) case [12] – a general proof for any gauge group can be found in Itzykson-Drouffe). Therefore, expectation values of quantities that are not gauge invariant must vanish, and we have to look for gauge-invariant correlation functions. In a similar way as in the Ising gauge theory, we consider a product along a closed, directed contour C containing angular variables:

$$\exp\left\{ i \sum_C \theta_\mu(n) \right\} . \tag{4.17}$$

In the same way as we have seen through eqs. (4.7) and (4.8), that $\theta_{\mu\nu}$ is gauge invariant, it can be seen that the directed sum above is also gauge invariant. The expectation value of the operator above leads to the so-called *Wilson-loop* [2]:

$$\left\langle \exp\left\{ i \sum_C \theta_\mu(n) \right\} \right\rangle = \frac{1}{Z} \int \mathcal{D}\theta_\mu \, \exp\left\{ i \sum_C \theta_\mu(n) \right\} \mathrm{e}^{-S} . \tag{4.18}$$

In the following we consider the behavior of the Wilson loop in the strong- and weak-coupling limits.

4.2.1 Strong-coupling limit

The strong coupling limit ($g \gg 1$) is equivalent to the high temperature limit in our discussion of the XY-model in Ch. 1. As in that case, for a large loop C, low orders of the expansion will lead to integrals over the angular variables, that vanish:

$$\int_0^{2\pi} \mathrm{d}\theta_\mu(n) \, \mathrm{e}^{i\theta_\mu(n)} = 0 . \tag{4.19}$$

Only when enough phase factors appear in the expansion, such that the phase factors of the correlation function are cancelled, a finite result will be achieved. To see this, we consider the expansion of the exponential of the action.

$$\begin{aligned} \exp\left\{ \frac{1}{2g^2} \sum_{n,\mu\nu} \cos\theta_{\mu\nu} \right\} &= \prod_{n,\mu\nu} \exp\left\{ \frac{1}{2g^2} \cos\theta_{\mu\nu} \right\} \\ &= \prod_{n,\mu\nu} \exp\left\{ \frac{1}{4g^2} \left[\mathrm{e}^{i\theta_{\mu\nu}} + \mathrm{e}^{-i\theta_{\mu\nu}} \right] \right\} \\ &= \prod_{n,\mu\nu} \sum_m \frac{1}{m!} \left\{ \frac{1}{4g^2} \left[\mathrm{e}^{i\sum_p \theta_\mu} + \mathrm{e}^{-i\sum_p \theta_\mu} \right] \right\}^m , \end{aligned} \tag{4.20}$$

where in the last line we introduced $\sum_p$ that denotes a directed sum of the angular variables along a plaquette. The non-vanishing contribution in lowest order will be given by the lowest power (i.e. $m = 1$) for each plaquette where one of the sides coincides with the loop C. The other sides of these plaquettes have to be compensated as well, so that finally, the minimal area subtended by C will be covered with plaquettes, as shown in Fig. 8. Hence, the number of plaquettes

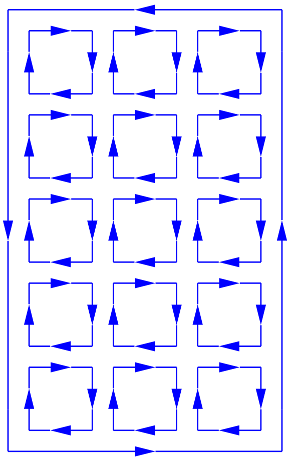

Figure 8: Plaquettes inside the contour C needed to cancel phase factors on C and its interior.

needed in order to obtain a non vanishing result is given by the minimal area A enclosed by C, such that,

$$\left\langle \exp\left\{ i \sum_C \theta_\mu(n) \right\} \right\rangle \simeq \left(\frac{1}{4g^2} \right)^A = \mathrm{e}^{-\ln(4g^2)A} \, . \tag{4.21}$$

As in the Ising case, we obtain the area law in the strong coupling (high temperature) limit. In general, assuming that the strong-coupling expansion has a finite radius of convergence, the result is

$$\left\langle \exp\left\{ i \sum_C \theta_\mu(n) \right\} \right\rangle = \mathrm{e}^{-f(g^2)A} \, , \tag{4.22}$$

where the function f is in leading order $\ln(4g^2)$.

4.2.2 Weak-coupling limit

In this case we have $g \ll 1$, i.e. the low temperature limit that we discussed briefly at the end of Sec. 4.1. Since we have seen there that a close connection can be established with electrodynamics in the continuous limit appropriate for $g \ll 1$, we return to eq. (4.15), and identify the Wilson loop in the form he originally introduced [2].

$$\begin{aligned} \left\langle \exp\left\{ i \sum_C \theta_\mu(n) \right\} \right\rangle &\to \left\langle \exp\left\{ ig \oint_C A_\mu \, \mathrm{d}x_\mu \right\} \right\rangle \\ &= \frac{1}{Z} \int \mathcal{D}A_\mu \exp\left(-\frac{1}{4a^{d-4}} \int \mathrm{d}^d x \, F_{\mu\nu}F_{\mu\nu} + ig \oint_C A_\mu \, \mathrm{d}x_\mu \right) \, , \end{aligned} \tag{4.23}$$

where since $A_\mu \sim \theta_\mu/a$, and $a \to 0$, the integral over A_μ goes from $-\infty$ to ∞. The limit $a \to 0$ does not present a problem in $d = 4$, where QED is renormalizable. Hence, from now on, we restrict ourselves to $d = 4$.

In order to proceed further, we have to fix the gauge in order to have a meaningfull evaluation of the integral (4.23). Since such an evaluation involves the propagator of the gauge fields, let us discuss briefly how it can be evaluated. Here we concentrate on the term with $F_{\mu\nu}^2$, that can be expressed as follows

$$S(A_\mu) = \frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F_{\mu\nu} \;=\; \frac{1}{2}\int \mathrm{d}^4k\, A_\mu(-k)\,\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right) A_\nu(k)\;, \tag{4.24}$$

where $k^2 = k_\mu k_\mu$. The propagator of the gauge fields is given by the inverse of the expression in brackets. However, it can be easily seen that unless we regularize the expression, the inverse does not exist. Hence, we introduce a mass for the gauge fields, that at the end will be sent to zero, i.e. we add a term

$$\frac{1}{2}m^2A_\mu^2 \tag{4.25}$$

to the action that leads to

$$(4.24) \to \frac{1}{2}\int \mathrm{d}^4k\, A_\mu \left[\left(k^2 + m^2\right)\delta_{\mu\nu} - k_\mu k_\nu\right] A_\nu\;. \tag{4.26}$$

Still we need to fix the gauge. For that purpose, we introduce a scalar field adding to the action a term

$$S(A_\mu,\chi) = S(A_\mu) - \frac{1}{2}\int \mathrm{d}^4x\left[(\partial_\mu\chi)^2 - \tilde{m}^2\chi^2\right]\;. \tag{4.27}$$

Since the term is bilinear in χ, this amounts to introducing a multiplicative constant to the partition function, that is cancelled by normalization in the correlation function. We can perform now a change of variables

$$A_\mu = A'_\mu + \frac{1}{m}\,\partial_\mu\chi\;, \tag{4.28}$$

that corresponds to a change of gauge. Therefore, $F_{\mu\nu}$ is not changed, so that the only change appears in the mass term.

$$\frac{1}{2}m^2A_\mu^2 = \frac{1}{2}m^2A'^2_\mu + mA'_\mu\partial_\mu\chi + \frac{1}{2}(\partial_\mu\chi)^2\;, \tag{4.29}$$

leading to

$$\begin{aligned} S(A'_\mu,\chi) &= S(A'_\mu) + \int \mathrm{d}^4x\left(mA'_\mu\partial_\mu\chi + \frac{1}{2}\tilde{m}^2\chi^2\right) \\ &= S(A'_\mu) - \int \mathrm{d}^4x\left(m\partial_\mu A'_\mu\,\chi - \frac{1}{2}\tilde{m}^2\chi^2\right)\;. \end{aligned} \tag{4.30}$$

At this point we can integrate over χ, since it appears in a bilinear form. We recall briefly in the following, how such an integration can be performed.

Gaussian integration

In order to discuss a Gaussian integration, we start by considering a Gaussian integral for a single degree of freedom

$$e^{\frac{1}{2}ax^2} = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} \mathrm{d}\phi \, e^{-\frac{\phi^2}{2a}+\phi x} \tag{4.31}$$

Next we consider a bilinear form with the result:

$$\begin{aligned} \exp\left(\frac{1}{2}\sum_{i,j}^{N} x_i \, A_{ij} \, x_j\right) &= \frac{1}{(2\pi)^{N/2}} \frac{1}{\sqrt{\det A}} \int \underbrace{\prod_{i=1}^{N} \mathrm{d}\phi_i}_{\mathcal{D}\phi} \\ &\quad \times \exp\left[-\frac{1}{2}\sum_{ij} \phi_i \, \left(A^{-1}\right)_{ij} \, \phi_j + \sum_i \phi_i x_i\right], \end{aligned} \tag{4.32}$$

with A a symmetric and positive definite matrix. In order to see that (4.32) follows from (4.31), we recall that since A is symmetric, there exists an orthogonal matrix M such that

$$M^T A M = A^D \quad \Rightarrow \quad A = M A^D M^T \, , \tag{4.33}$$

where $\left(A^D\right)_{ij} = a_i \delta_{ij}$ is a diagonal matrix. In the same way we can define transformed vectors $\boldsymbol{x}' = M^T \boldsymbol{x}$, such that

$$\begin{aligned} \exp\left(\frac{1}{2}\,\boldsymbol{x}^T \, A \, \boldsymbol{x}\right) &= \exp\left(\frac{1}{2}\boldsymbol{x}'^T A^D \boldsymbol{x}'\right) \\ &= \exp\left(\frac{1}{2}\sum_i x_i' \, a_i \, x_i'\right) \\ &= \prod_i \exp\left(\frac{1}{2}\, x_i' \, a_i \, x_i'\right) \, . \end{aligned} \tag{4.34}$$

For each term in the product (4.34), we can apply the identity (4.31), leading to

$$\prod_i \exp\left(\frac{1}{2}\, x_i' \, a_i \, x_i'\right) = \prod_i \frac{1}{\sqrt{2\pi a_i}} \int_{-\infty}^{\infty} \mathrm{d}\phi_i' \, \exp\left(-\frac{\phi_i'^2}{2a_i} + \phi_i' x_i'\right) \, , \tag{4.35}$$

with $\boldsymbol{\phi}' = M\boldsymbol{\phi}$. Undoing the orthogonal transformation we arrive at the result (4.32).

Once we have recapitulated how a Gaussian integral is performed, we apply the above to the integral over χ, leading to

$$\tilde{S}(A_\mu') = S(A_\mu') + \frac{m^2}{2\tilde{m}^2} \int \mathrm{d}^4x \, \left(\partial_\mu A_\mu'\right)^2 \, . \tag{4.36}$$

A choice of gauge corresponds now to a choice of $\tilde{m}$. Taking $\tilde{m} = m$, we have the Feynman gauge. As we will see below, after such a choice, the propagator becomes particularly simple. However, since it is the first time that we consider the propagator of a field theory, let us recall how to obtain it.

As propagator we understand an expectation value of the corresponding fields propagating from a point to another (let us say from 0 to x):

$$\langle A_\mu(x) A_\mu(0) \rangle \quad = \quad \frac{1}{Z} \int \mathcal{D} A_\mu \, A_\mu(x) \, A_\mu(0) \, \mathrm{e}^{-S(A)} \, . \tag{4.37}$$

The propagator can be obtained from the generating functional (i.e. in our case the partition function) by introducing sources that couple to the fields of interest, and differentiating with respect to the sources. In our case, we introduce sources coupling to the gauge-fields, such that the generating functional is as follows:

$$Z\left[J\right] \quad = \quad \int \mathcal{D} A_\mu \, \exp \left[-S(A) + \int \mathrm{d}^d x \, J_\mu(x) \, A_\mu(x) \right] \, . \tag{4.38}$$

The propagator is then obtained by differentiating with respect to J at the points x and 0:

$$\langle A_\mu(x) A_\mu(0) \rangle \quad = \quad \frac{\delta^2 \ln Z[J]}{\delta J_\mu(x) \, \delta J_\mu(0)} \, . \tag{4.39}$$

Using the action obtained in (4.30), i.e.

$$S(A_\mu) \quad = \quad \frac{1}{2} \int \mathrm{d}^4 k \, A_\mu(-k) \, \left(k^2 + m^2\right) A_\mu(k) \, , \tag{4.40}$$

that is quadratic in A_μ, we can perform the integration over A_μ, since it is a Gaussian integration. With the notation

$$G_0^{-1}(k) \quad = \quad k^2 + m^2 \, , \tag{4.41}$$

we have after performing the Gaussian integration

$$Z[J] = Z[0] \, \mathrm{e}^{W[J]} \, , \tag{4.42}$$

where

$$W[J] \quad = \quad \frac{1}{2} \int \mathrm{d}^d k J(-k) \, G_0(k) J(k) \, , \tag{4.43}$$

such that for the propagator of the gauge field we obtain

$$\langle A_\mu A_\nu \rangle(k) = \frac{\delta_{\mu\nu}}{k^2 + m^2} \, . \tag{4.44}$$

From there, and assuming that since we are on a lattice, we do not need to fear ultraviolet divergences, we can Fourier-transform the result above in order to finally obtain the form of the propagator in real space.

Since we have to perform a Fourier-transform in $d = 4$ of a rotational invariant function, we recall the form of the Jacobian when going to spherical coordinates in d dimensions:

$$\mathrm{d}^d k \;=\; k^{d-1}\mathrm{d}k\,\mathrm{d}\phi\,\sin\theta_1\mathrm{d}\theta_1\,\sin^2\theta_2\mathrm{d}\theta_2\cdots\sin^{d-2}\theta_{d-2}\mathrm{d}\theta_{d-2}\;. \tag{4.45}$$

For completeness, we recall the form of the area of a unit sphere in d dimensions:

$$S_d \;=\; \int_0^{2\pi}\mathrm{d}\phi\left[\prod_{j=1}^{d-2}\int_0^{\pi}\sin^j\theta_j\,\mathrm{d}\theta_j\right] = \frac{2\pi^{d/2}}{\Gamma\left(d/2\right)}\;, \tag{4.46}$$

where $\Gamma(x)$ is the gamma function. Going back to our case, the Fourier-transform of (4.44) is given by

$$\begin{aligned}\langle A_\mu(x)A_\nu(0)\rangle \;&=\; 2\pi\int_0^\infty\frac{\mathrm{d}k}{(2\pi)^4}\,\frac{k^3}{k^2+m^2}\int_0^\pi\mathrm{d}\theta_1\,\sin\theta_1\,\mathrm{e}^{ik|x|\cos\theta_1}\int_0^\pi\mathrm{d}\theta_2\sin^2\theta_2\\ &=\;\frac{1}{8\pi^2\mid x\mid}\int_0^\infty\mathrm{d}k\,\frac{k^2}{k^2+m^2}\sin\left(k\mid x\mid\right)\;.\end{aligned} \tag{4.47}$$

The resulting integral is safely convergent in the IR, so that we can take $m \to 0$. However, we have to regularize it in the UV. We do so with an exponential cutoff.

$$(4.47) \to \frac{1}{8\pi^2\mid x\mid}\int_0^\infty\mathrm{d}k\,\sin\left(k\mid x\mid\right)\,\mathrm{e}^{-ak} = \frac{1}{8\pi^2\left(a^2+\mid x\mid^2\right)}\;. \tag{4.48}$$

Hence, we can write

$$\langle A_\mu(x)A_\nu(0)\rangle \;=\; \delta_{\mu\nu}\,\Delta(x)\;, \tag{4.49}$$

with

$$\Delta(x-y) \;=\; \Delta(0)\delta_{x,y} + \Delta'(x-y)\;, \tag{4.50}$$

where

$$\Delta'(x-y) \;=\; \begin{cases} 1/\left(8\pi^2\mid x-y\mid\right) & \text{if}\quad \mid x-y\mid > a\;,\\ 0 & \text{otherwise}\;.\end{cases} \tag{4.51}$$

Now we are able to evaluate (4.23) since it is again a Gaussian integral. The result of the evaluation is

$$\begin{aligned}\left\langle\exp\left\{ig\oint_C A_\mu\,\mathrm{d}x_\mu\right\}\right\rangle \;&=\; \exp\left[-\frac{1}{2}g^2\oint_C\oint_C\langle A_\mu(x)A_\nu(y)\rangle\mathrm{d}x_\mu\mathrm{d}x_\nu\right]\\ &=\; \exp\left[-\frac{1}{2}g^2\oint_C\oint_C\Delta(x-y)\mathrm{d}x_\mu\mathrm{d}y_\mu\right]\;.\end{aligned} \tag{4.52}$$

Before going into an explicit evaluation of the integral, we can first try to assess the outcome of it. The propagator of the gauge fields corresponds to an interaction, that in this case takes place between line elements $\mathrm{d}x_\mu$ and $\mathrm{d}y_\mu$, and is given by an exchange of virtual photons. In the case of a large loop, the dependence of the Wilson loop on the dimensions of the loop will reflect the long-distance behavior of the propagator. If the propagator is short ranged, i.e. it decays exponentially within a distance ξ, then, the correlator (4.52) will be determined by the contributions of $\mathrm{d}x_\mu$ and $\mathrm{d}y_\mu$ within the distance ξ. For $C \gg \xi$, the correlation function should then obey the perimeter law. In the U(1) case we discuss here, we have seen that $\Delta(x-y)$ decays actually with a power law. We will discuss its consequences in an explicit calculation below.

In order to evaluate (4.52), we consider a rectangular contour as in Fig. 8. We

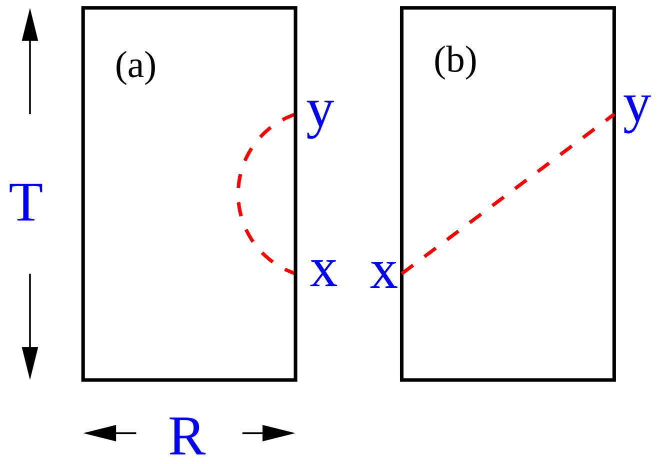

Figure 9: Contributions to the Wilson loop.

separate the calculation of the integral for the Wilson loop into two contributions as shown in Fig. 9. On the one hand, we have contributions where an edge interacts with itself, as shown in Fig. 9 (a). On the other hand, two different edges can interact with each other. Notice that orthogonal edges do not interact since in that case, $\mathrm{d}x_\mu \mathrm{d}y_\mu = 0$. The integral corresponding to Fig. 9 (a) gives

$$\begin{aligned}\int\int \Delta'(x-y)\mathrm{d}x_\mu \mathrm{d}y_\mu &= \frac{2}{8\pi^2}\int_a^T \mathrm{d}y \int_0^{y-a} \mathrm{d}x \, \frac{1}{(y-x)^2} \\ &= \frac{1}{4\pi^2}\left[\frac{T}{a} - \ln\left(\frac{T}{a}\right)\right] \, , \qquad (4.53)\end{aligned}$$

when going along the side T from bottom to top, and the same contribution is obtained when going from top to bottom. The factor 2 comes from the two possibilities, i.e. $x > y$ and $x < y$. When integrating along the side R, the result is

$$\begin{aligned}\int\int \Delta'(x-y)\mathrm{d}x_\mu \mathrm{d}y_\mu &= \frac{2}{8\pi^2}\int_a^R \mathrm{d}y \int_0^{y-a} \mathrm{d}x \, \frac{1}{(y-x)^2} \\ &= \frac{1}{4\pi^2}\left[\frac{R}{a} - \ln\left(\frac{R}{a}\right)\right] \, . \qquad (4.54)\end{aligned}$$

Finally, the total contribution to Wilson's loop from Fig. 9 (a) is

$$\oint_{C(a)} \oint_{C(a)} \Delta'(x-y) \mathrm{d}x_\mu \mathrm{d}y_\mu = \frac{1}{2\pi^2} \left[\frac{T+R}{a} - \ln\left(\frac{T}{a}\right) - \ln\left(\frac{R}{a}\right) \right] . \quad (4.55)$$

The integral corresponding to Fig. 9 (b) gives

$$\begin{aligned} \int\int \Delta'(x-y) \mathrm{d}x_\mu \mathrm{d}y_\mu &= -\frac{2}{8\pi^2} \int_0^T \mathrm{d}y \int_0^T \mathrm{d}x \, \frac{1}{R^2 + (y-x)^2} \\ &= -\frac{1}{4\pi^2} \left[2\frac{T}{R} \arctan\left(\frac{T}{R}\right) - \ln\left(\frac{R^2+T^2}{R^2}\right) \right] . \quad (4.56) \end{aligned}$$

when going along the side T. For the side R, we need just to exchange T and R:

$$\begin{aligned} \int\int \Delta'(x-y) \mathrm{d}x_\mu \mathrm{d}y_\mu &= -\frac{2}{8\pi^2} \int_0^R \mathrm{d}y \int_0^R \mathrm{d}x \, \frac{1}{T^2 + (y-x)^2} \\ &= -\frac{1}{4\pi^2} \left[2\frac{R}{T} \arctan\left(\frac{R}{T}\right) - \ln\left(\frac{R^2+T^2}{T^2}\right) \right] . \quad (4.57) \end{aligned}$$

Before obtaining the complete expression for Wilson's loop, we can simplify our task by considering the case where $T \gg R$, i.e. anticipating the limit of zero temperature for the corresponding quantum theory that we could reach using the transfer matrix. In this case, the last integral leads to a vanishing expression while for (4.56) we have

$$(4.56) \simeq -\frac{1}{4\pi}\frac{T}{R} + \frac{1}{2\pi^2} \ln\left(\frac{T}{a}\right) - \frac{1}{2\pi^2} \ln\left(\frac{R}{a}\right) , \quad (4.58)$$

such that finally,

$$\oint_C \oint_C \Delta(x-y) \mathrm{d}x_\mu \mathrm{d}y_\mu = \left(\Delta(0) + \frac{1}{4\pi^2 a} \right) P - \frac{1}{4\pi}\frac{T}{R} - \frac{1}{\pi^2} \ln\left(\frac{R}{a}\right) , \quad (4.59)$$

where $P = 2(T+R)$ is the perimeter of C. The final result for the correlation function is

$$\left\langle \exp\left\{ ig \oint_C A_\mu \, \mathrm{d}x_\mu \right\} \right\rangle = \exp\left[-\frac{1}{2} g^2 c P + \frac{g^2}{8\pi}\frac{T}{R} + \frac{g^2}{2\pi^2} \ln\left(\frac{R}{a}\right) \right] , \quad (4.60)$$

where c is a constant depending on the lattice propagator at the origin, and in general of the UV cutoff. Here we obtained a result corresponding to a perimeter law but with corrections due to the long-range character of the propagator. In any case, the behavior of the correlation function is qualitatively different from the one in strong-coupling, strongly suggesting that the Abelian lattice gauge theory has two different phases.

4.2.3 Potential for static charges and confinement

A further understanding of the results obtained for the correlation function can be gained by recalling that in electrodynamics, that based on our weak-coupling analysis should correspond to that limit of the theory, the gauge fields mediate an interaction among charges in the system. In particular, for static charges, we should have Coulomb's law.

In order to see that this is the case, we recall the form of the coupling of the electromagnetic fields to currents:

$$e \int A_\mu(x) J_\mu(x) \, \mathrm{d}^4x \ , \tag{4.61}$$

a term that should be added to the action. Furthermore, since the current is a conserved quantity, we can consider it on a closed loop. Let the loop be C, the same rectangular loop we were considering in our discussion of the strong- and weak-coupling limits. Since the loop has infinitesimal cross-section, we are actually considering not a current density but a current on it. Hence, J_μ can be taken to be unity on the directed loop and zero elsewhere. Then, for a given time slice at time τ, we have $J_0 = -1$ at $x = 0$, and $J_0 = +1$ at $x = R$. From the point of view of the transfer matrix, we have a system with a static charge at $x = R$ and its antiparticle at $x = 0$. With such a picture in mind, the expectation value of the Wilson loop can be viewed as the ratio of the partition function of the system coupling to external charges (i.e. with an action including (4.61)) to the partition function of the systems without them:

$$\left\langle \exp \left\{ ig \oint_C A_\mu \, \mathrm{d}x_\mu \right\} \right\rangle \quad = \quad \frac{Z[J]}{Z[0]} \ . \tag{4.62}$$

This can be also expressed in terms of the free energy of the system since $\mathcal{F} = -\ln Z$, where we have absorbed the factor $1/k_B T$ in the definition of $\mathcal{F}$ such that

$$\frac{Z[J]}{Z[0]} \quad = \quad \exp\left\{ -\left[\mathcal{F}(J) - \mathcal{F}(0) \right] \right\} \ . \tag{4.63}$$

In the limit $T \to \infty$ (i.e. when the temporal side of C becomes very long), we approach the ground-state of the equivalent Hamiltonian and the difference in free energies, since they are extensive quantities, should obey

$$\mathcal{F}(J) - \mathcal{F}(0) \propto T \ . \tag{4.64}$$

The proportionality constant is the energy difference between the ground-state of the Hamiltonian with and without charges. For static charges this difference is purely due to the potential energy, such that

$$\mathcal{F}(J) - \mathcal{F}(0) = V(R)\, T \ . \tag{4.65}$$

Putting the expressions above together, we have

$$\left\langle \exp\left\{ ig \oint_C A_\mu \, \mathrm{d}x_\mu \right\} \right\rangle \quad = \quad \mathrm{e}^{-V(R)T} \;, \tag{4.66}$$

or equivalently,

$$V(R) \quad = \quad -\lim_{T\to\infty} \frac{1}{T} \ln \left\langle \exp\left\{ ig \oint_C A_\mu \, \mathrm{d}x_\mu \right\} \right\rangle \;, \tag{4.67}$$

the potential acting on static charges hold a distance R appart.

We can now go back to our result (4.60). There, the perimeter law would lead to a constant potential, meaning that the charges are free. However, taking into account the second term leads to

$$V(R) \sim \text{const.} - \frac{g^2}{8\pi} \frac{1}{R} \;, \tag{4.68}$$

that corresponds to Coulomb's law. Hence, we can reproduce in this limit the well known result $V(R) \sim e^2/R$. We can also give a meaning to the result obtained in the strong-coupling limit, where the area law leads to

$$V(R) \sim \mid R \mid \;, \tag{4.69}$$

i.e. we obtain *confinement* of particle and antiparticle. While in the traditional frame of a quantum field theory, no way is known up to now to reach confinement, it appears naturally in a lattice gauge theory in the strong coupling limit.

4.3 Two-dimensional Abelian lattice gauge theory

As we have seen in the case of the Ising lattice gauge theory, it can be expected that as we lower the dimensions, only one phase may subsist for all non-zero values of the coupling constant. In order to see this, we perform a direct calculation.

As in the case of the Ising lattice gauge theory, it is convenient to choose a definite gauge. Here we take the temporal gauge with

$$\theta_0(n) = 0 \;, \tag{4.70}$$

such that $\exp\left[i\theta_\mu(n)\right] = 1$ for μ in the τ-direction. Choosing a rectangular contour, $\theta_\mu(n) \neq 0$ only on its horizontal links, i.e. the remaining variable is $\theta_1(n)$. Then, the correlation function is given by

$$\left\langle \exp\left(i \sum_C \theta_\mu \right) \right\rangle \quad = \quad \frac{1}{Z} \int \prod \mathrm{d}\theta_1(n) \, \exp\left[\beta \sum_{n,\mu\nu} \cos\theta_{\mu\nu} + i \sum_C \theta_\mu \right] \;, \tag{4.71}$$

where C is a closed contour, and $\beta \equiv 1/2g^2$. In order to see that it is possible to perform a change of variables from θ_μ to $\theta_{\mu\nu}$, such that a manifestly gauge-invariant form can be obtained for the evaluation of the correlation function, we notice first, that

$$\sum_C \theta_\mu = \sum_{\{P_C\}} \theta_{\mu\nu}(n) \; , \tag{4.72}$$

where $\{P_C\}$ denotes the set of plaquettes enclosed within C. This can be obtained directly from the definition (4.6) together with the convention (4.5). Equation (4.72) is the lattice version of Stoke's law. In order to change variables, we need the Jacobian of the transformation. To this end, we notice that in two dimensions, $\theta_{\mu\nu}$ can be determined in terms of e.g. θ_{01}. From the definition (4.6), and taking the temporal gauge into account, we have

$$\theta_{10}(n) = -\theta_1(n+\hat{\tau}) + \theta_1(n) \; , \tag{4.73}$$

that leads to

$$\begin{aligned} \theta_1(n) &= -\theta_{10}(n-\hat{\tau}) + \theta_1(n-\hat{\tau}) \\ &= -\theta_{10}(n-\hat{\tau}) - \theta_{10}(n-2\hat{\tau}) + \theta_1(n-2\hat{\tau}) \\ &= -\sum_{\tau'<\tau} \theta_{\mu\nu}(\tau', x) \; , \end{aligned} \tag{4.74}$$

such that the Jacobian is one. Finally, after the change of variables we have

$$\begin{aligned} &\left\langle \exp\left(i \sum_C \theta_\mu \right) \right\rangle \\ &= \frac{\int \prod_{\{P_C\}} \mathrm{d}\theta_{\mu\nu}(n) \, \exp\left[\beta \sum_{n,\mu\nu} \cos\theta_{\mu\nu} + i \sum_{P_C} \theta_{\mu\nu}\right]}{\int \prod_{\{P_C\}} \mathrm{d}\theta_{\mu\nu}(n) \, \exp\left[\beta \sum_{n,\mu\nu} \cos\theta_{\mu\nu}\right]} \; , \end{aligned} \tag{4.75}$$

where common factors from numerator and denominator were cancelled. The expression above corresponds to a product of independent integrations, with the plaquettes being thus decoupled, such that

$$(4.75) = \left\{ \frac{\int_0^{2\pi} \mathrm{d}\theta_{\mu\nu} \, \exp\left[\beta\cos\theta_{\mu\nu} + i\theta_{\mu\nu}\right]}{\int_0^{2\pi} \mathrm{d}\theta_{\mu\nu} \, \exp\left[\beta\cos\theta_{\mu\nu}\right]} \right\}^A \; , \tag{4.76}$$

where A is the number of plaquettes enclosed by C. For the integrals in the expression above we have

$$\begin{aligned} \int_0^{2\pi} \mathrm{d}\theta_{\mu\nu} \, \exp\left[\beta\cos\theta_{\mu\nu}\right] &= 2\pi I_0(\beta) \; , \\ \int_0^{2\pi} \mathrm{d}\theta_{\mu\nu} \, \exp\left[\beta\cos\theta_{\mu\nu} + i\theta_{\mu\nu}\right] &= 2\pi I_1(\beta) \; , \end{aligned} \tag{4.77}$$

where $I_0(z)$ and $I_1(z)$ are incomplete Bessel functions, such that

$$\left\langle \exp\left(i \oint_C \theta_\mu\right) \right\rangle = \left[\frac{I_1(\beta)}{I_0(\beta)}\right]^A . \tag{4.78}$$

With $\beta = 1/2g^2$, we can obtain the results for the strong ($g^2 \gg 1 \leftrightarrow \beta \ll 1$) and weak-coupling ($g^2 \ll 1$) limits. In strong coupling we have

$$\begin{aligned} I_0(z) &= 1 + \mathcal{O}\left(z^2\right) , \\ I_1(z) &= \frac{z}{2} + \mathcal{O}\left(z^3\right) , \end{aligned} \tag{4.79}$$

such that

$$\frac{I_1(\beta)}{I_0(\beta)} \simeq \frac{\beta}{2} = \frac{1}{4g^2} , \tag{4.80}$$

leading to

$$\left\langle \exp\left(i \oint_C \theta_\mu\right) \right\rangle \simeq \mathrm{e}^{-\ln\left(4g^2\right)A} . \tag{4.81}$$

This is the same result as the one obtained in (4.21). On the other hand, in the weak-coupling limit we have the asymptotic expression for the incomplete Bessel functions

$$I_\nu(z) = \frac{\mathrm{e}^z}{\sqrt{2\pi z}} \left(1 - \frac{4\nu^2 - 1}{8z} + \cdots\right) , \tag{4.82}$$

such that

$$\frac{I_1(\beta)}{I_0(\beta)} \simeq 1 - \frac{1}{2\beta} = 1 - g^2 , \tag{4.83}$$

that leads to

$$\left\langle \exp\left(i \oint_C \theta_\mu\right) \right\rangle \simeq \mathrm{e}^{\ln\left(1-g^2\right)A} \simeq \mathrm{e}^{-g^2 A} . \tag{4.84}$$

We can see explicitely, that in the two-dimensional case, we are allways in the confining phase with the interquark potential varying smoothly between $g^2 \mid R \mid$ at weak coupling to $\ln(4g^2) \mid R \mid$ at strong coupling.

It should be noted that the construction used here relies on the fact that the plaquettes variables can be considered as independent. However, e.g. in three dimensions, an elementary volumen element coresponds to a cube with six plaquettes. The oritentation for the circulation can be chosen in such a way that

$$\sum_{\text{faces}} \theta_{\mu\nu} = 0 . \tag{4.85}$$

Such a constraint, that corresponds to a lattice version of Gauss' law, shows that the plaquettes cannot be treated as independent.

4.4 The quantum Hamiltonian formulation and quark confinement

We can learn more about the Abelian lattice gauge theory by considering the corresponding quantum Hamiltonian. As we have already seen in the case of the Ising lattice gauge theory, the first step is to allow for different couplings along the time and spatial directions. Here we can write

$$S = \beta_\tau \sum_{n,k} [1 - \cos\theta_{0k}(n)] - \beta \sum_{n,ik} \cos\theta_{ik}(n) \ , \tag{4.86}$$

where the temporal index is 0 and latin letters denote spatial links. As before, we take the temporal gauge, i.e. $\theta_0(n) = 0$. In this gauge there are still τ-independent gauge transformations under which the system is invariant.

In the temporal gauge, we have

$$\theta_{0k} = \theta_k(n + \hat{\tau}) - \theta_k(n) \ . \tag{4.87}$$

Since, as in the Ising case, we are interested in the limit $\beta_\tau \to \infty$, i.e. we are interested in the thermodynamic limit of the original model, only slowly varying configurations of θ_k will be important Therefore, we can approximate the first term in (4.86) as follows:

$$1 - \cos\theta_{0k} \simeq \frac{1}{2}\theta_{0k}^2 \simeq \frac{1}{2}a_\tau^2 \left(\frac{\partial\theta_k}{\partial\tau}\right)^2 \ , \tag{4.88}$$

where a_τ denotes the lattice constant in the temporal direction. Going over to a τ-continuum, we can replace the sums over sites along the temporal direction by integrals:

$$\sum_{n,k} \to \frac{1}{a_\tau}\int \mathrm{d}\tau \sum_{\boldsymbol{n},k} \ , \tag{4.89}$$

where $\boldsymbol{n}$ denotes the sites in the spatial directions. With all these changes, the action becomes

$$S = \int \mathrm{d}\tau \left\{ \frac{1}{2}\beta_\tau a_\tau \sum_{\boldsymbol{n},k} \left(\frac{\partial\theta_k}{\partial\tau}\right)^2 - \frac{\beta}{a_\tau}\sum_{\boldsymbol{n},ik} \cos\theta_{ik}(\tau,\boldsymbol{n}) \right\} \ . \tag{4.90}$$

We are interested in the limits

$$\begin{aligned} \beta_\tau &= \frac{g^2}{a_\tau} \to \infty \ , \\ \beta &= \frac{a_\tau}{g^2} \to 0 \ , \end{aligned} \tag{4.91}$$

where g^2 is held finite at a given fixed value. This is in fact the same limit we took when discussing the transfer matrix of the Ising model in Sec. 2.2.

In order to define a quantum Hamiltonian, a Hilbert space has to be set up for each τ-surface. In our case, the quantum field is given by $\theta_k(\boldsymbol{n})$. We define a canonically conjugate momentum density $L_k(\boldsymbol{n})$ with the commutation relation

$$[\theta_k(\boldsymbol{n}'), L_i(\boldsymbol{n})] = i\, \delta_{ik}\, \delta_{\boldsymbol{n}\boldsymbol{n}'} \; . \tag{4.92}$$

Since θ_k is a planar angle, L_k is an angular momentum with eigenvalues $m \in \mathbb{Z}$. At this point we recall what happens with other familiar canonically conjugated variables, namely x and p. In that case, it can be shown (e.g. when going to a path integral) that

$$\langle x_{i+1} | \mathrm{e}^{-\epsilon \hat{p}^2/2} | x_i \rangle \propto \exp\left[-\frac{1}{2}\epsilon \left(\frac{x_{i+1} - x_i}{\epsilon} \right)^2 \right] \; . \tag{4.93}$$

That is, the square of the time derivative goes over into the square of the conjugate operator in the exponential. We expect something similar to happen in this case, although we are dealing now with a discrete spectrum. However, in the same way as it is shown in elementary quantum mechanics for x and p, we can see here that, using the fact that for each operator θ_k there is an eigenstate fulfilling

$$\theta_k \mid \theta \rangle \;\; = \;\; \theta \mid \theta \rangle \; , \tag{4.94}$$

where

$$\langle \theta' \mid \theta \rangle = \delta(\theta' - \theta) \; , \tag{4.95}$$

then,

$$\langle \theta \mid \ell \rangle \;\; \propto \;\; \mathrm{e}^{i\ell\theta} \; , \tag{4.96}$$

where $\mid \ell \rangle$ is an eigenstate of L_k. Using the above, we have in our case

$$\begin{aligned} \langle \theta' \mid \mathrm{e}^{-\epsilon L^2/2} \mid \theta \rangle &= \sum_{\ell,\ell'} \langle \theta' \mid \ell' \rangle \langle \ell' \mid \mathrm{e}^{-\epsilon L^2/2} \mid \ell \rangle \langle \ell \mid \theta \rangle \\ &= \sum_{\ell} \exp\left[-\frac{1}{2}\epsilon \ell^2 + i\ell(\theta' - \theta) \right] \; . \end{aligned} \tag{4.97}$$

Here we can use the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) \;\; = \;\; \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) \; , \tag{4.98}$$

where $f(x)$ is a continuous function with Fourier-transform $\hat{f}(q)$. In our case, this translates into

$$\sum_{\ell} \exp\left[-\frac{1}{2}\epsilon \ell^2 + i\ell(\theta' - \theta) \right] \;\; = \;\; \sqrt{\frac{2\pi}{\epsilon}} \sum_{k} \exp\left[-\frac{1}{2\epsilon}(\theta' - \theta - 2\pi k)^2 \right] \; . \tag{4.99}$$

Since we are interested in the limit $\epsilon \to 0$, and we assume that θ' and θ are close to each other, only the term with $k = 0$ remains. In this way, the Hamiltonian resulting from the transfer matrix has the form

$$aH = \frac{1}{2} g^2 \sum_{\boldsymbol{n},k} L_k^2(\boldsymbol{n}) - \frac{1}{g^2} \sum_{\boldsymbol{n},ik} \cos\theta_{ik}(\tau, \boldsymbol{n}) \;, \tag{4.100}$$

where a is a lattice constant introduced in order to have the proper units for H.

Once we have obtained the Hamiltonian of the system, we can consider its symmetries, in particular its invariance under τ-independent local gauge transformations. Having seen that an angular momentum operator appears in the theory in a natural way, we can use it to perform rotations of the variables on the links ℓ connecting to a site $\boldsymbol{n}$ by defining

$$G_\chi(\boldsymbol{n}) = \exp\left[i \sum_{\pm\ell} L_\ell(\boldsymbol{n})\, \chi \right] . \tag{4.101}$$

This can be generalized to local gauge transformations acting on all sites as follows

$$G_\chi = \exp\left[i \sum_{\boldsymbol{n},\ell} L_\ell(\boldsymbol{n})\, \chi(\boldsymbol{n}) \right] . \tag{4.102}$$

It can be easily verified that

$$G_\chi\, \theta_k\, G_\chi^{-1} = \theta_k + \chi(\boldsymbol{n}) - \chi(\boldsymbol{n}+k) = \theta_k - \Delta_k \chi \;, \tag{4.103}$$

that corresponds to a τ-independent gauge transformation. Since the Hamiltonian (4.100) is gauge invariant, it holds that

$$G_\chi\, H\, G_\chi^{-1} = H \;. \tag{4.104}$$

Furthermore, due to Elitzur's theorem, the states in the physical space are also gauge invariant, such that

$$G_\chi \mid \theta\rangle = \mid \theta\rangle \;. \tag{4.105}$$

At this point we recall the connection between the gauge fields in the lattice gauge theory and the corresponding ones in electrodynamics that was given in (4.15). With it we can rewrite the commutation relations (4.92) as follows.

$$\frac{g}{a^2} \left[A_k(\boldsymbol{n}'), L_i(\boldsymbol{n}) \right] = i\, \delta_{ik}\, \frac{1}{a^3} \delta_{\boldsymbol{n}\boldsymbol{n}'} \;. \tag{4.106}$$

Identifying $\delta_{\boldsymbol{n}\boldsymbol{n}'}/a^3$ with the discrete form of the Dirac delta function, we can view the commutator above as the commutation relation of quantum electrodynamics

$$\left[A_i(\boldsymbol{x}), E_j(\boldsymbol{x}') \right] = i \delta_{ij}\, \delta\left(\boldsymbol{x} - \boldsymbol{x}' \right) \;, \tag{4.107}$$

where $\boldsymbol{E}(\boldsymbol{x})$ is the electric field. Without reviewing here quantum electrodynamics, let us give a simple argument, as to why the electric field is canonical conjugate to A_μ. For that purpose let us look at the classical field equations of electrodynamics:

$$\partial_\mu F^{\mu\nu} = j^\nu \ , \tag{4.108}$$

in appropriate units. We can expresse this equation for a particular gauge, the one with $A_0 = 0$, that is equivalent to our temporal gauge. In this case, the equations of motion become

$$\begin{aligned} \partial_j \dot{A}_j &= j_0 \ , \\ \ddot{A}_i - \partial_j F_{ji} &= j_i \ , \end{aligned} \tag{4.109}$$

where the indices are treated now in Euclidean space. The classical Lagrangean leading to such equations of motion is

$$L(A_i) = \int \mathrm{d}^{d-1}x \left[\frac{1}{2}\dot{A}_i^2 - \frac{1}{4}F_{ij}^2 + j_i A_i \right] . \tag{4.110}$$

The canonical momentum is given in this case by

$$\Pi_i(\boldsymbol{x}, t) = \dot{A}_i(\boldsymbol{x}, t) \ , \tag{4.111}$$

that corresponds to the electric field. We see then, that the electric field can be related to the angular momentum as follows.

$$E_i(\boldsymbol{n}) = \frac{g}{a^2} L_i(\boldsymbol{n}) \ . \tag{4.112}$$

As we mentioned before, since L_i has a discrete spectrum with eigenvalues $m \in \mathbb{Z}$, the equation above implies that the electric flux on a link , $a^2 E_i(\boldsymbol{n})$, is quantized in units of charge g. The quantization is a consequence of considering $A_k(\boldsymbol{n})$ an angular variable, and hence, this situation is often referred to as *compact QED* [13].

We can use (4.112) in order to express the Hamiltonian (4.100) in terms of an electric field,

$$H = \frac{1}{2} \sum_{\boldsymbol{n},k} a^3 E_k^2(\boldsymbol{n}) - \frac{1}{g^2 a} \sum_{\boldsymbol{n},ik} \cos\theta_{ik}(\tau, \boldsymbol{n}) \ , \tag{4.113}$$

where we see now that the first term is the discrete form of the contribution to the energy by the electric field. Hence, we should expect that the second term is the magnetic contribution to the energy. In fact, such an identification can be made since in electrodynamics we have

$$B_i = \varepsilon_{ijk} \partial_j A_k \ , \tag{4.114}$$

so that we can set

$$\theta_{jk} = a^2 g \, \varepsilon_{ijk} B_i \ . \tag{4.115}$$

Then, in the continuum limit, we have

$$\begin{aligned} -\frac{1}{g^2 a}\sum_{\boldsymbol{n},ik} \cos\theta_{ik}(\tau,\boldsymbol{n}) &\simeq -\frac{1}{g^2 a}\sum_{\boldsymbol{n},ik}\left(1-\frac{1}{2}\theta_{ik}^2\right) \\ &\simeq \frac{1}{2}\sum_{\boldsymbol{n},k} a^3 B_k^2(\boldsymbol{n}) + \text{const.}\ , \end{aligned} \tag{4.116}$$

as expected.

In order to relate the behavior of the Hamiltonian (4.113) to the previously discussed phases, and in particular, in order to see whether confinement is present or not, we need to stipulate how static charges can be introduced in the theory. We introduce additional angular variables $\theta(\boldsymbol{n})$ on the sites of the model, and consequently, a momentum $L(\boldsymbol{n})$ conjugate to $\theta(\boldsymbol{n})$, obeying

$$[\theta(\boldsymbol{n}), L(\boldsymbol{n}')] = i\,\delta_{\boldsymbol{n}\boldsymbol{n}'}\ . \tag{4.117}$$

The phase variables $\theta(\boldsymbol{n})$ exponentiated will be used to describe matter fields, that is they are given by

$$\psi(\boldsymbol{n}) = \mathrm{e}^{\pm i\theta(\boldsymbol{n})}\ , \tag{4.118}$$

and the generators of local gauge transformations at a site $\boldsymbol{n}$ will be given now by

$$\sum_{\pm j} L_j(\boldsymbol{n}) + L(\boldsymbol{n})\ , \tag{4.119}$$

with a generalization to the whole lattice given by

$$G_\chi = \exp\left[i\sum_{\boldsymbol{n},\ell} L_\ell(\boldsymbol{n})\,\chi(\boldsymbol{n}) + i\sum_{\boldsymbol{n}} L(\boldsymbol{n})\,\chi(\boldsymbol{n})\right]\ . \tag{4.120}$$

Then, a gauge transformation of the matter field is performed as follows

$$G_\chi\,\psi(\boldsymbol{n})G_\chi^{-1} = G_\chi\,\mathrm{e}^{\pm i\theta(\boldsymbol{n})}G_\chi^{-1} = \mathrm{e}^{\pm i\chi(\boldsymbol{n})}\mathrm{e}^{\pm i\theta(\boldsymbol{n})}\ , \tag{4.121}$$

that is, the gauge transformation corresponding to a charged field with $\pm g$ units of charge.

Once we have specified, how the matter field looks like, we can consider an operator to create a state with static charges $\pm g$ at points, say zero and $\boldsymbol{R}$. While

$$\mathrm{e}^{-i\theta(\boldsymbol{0})}\,\mathrm{e}^{i\theta(\boldsymbol{R})} \tag{4.122}$$

appears as a first possibility, we notice that this is not gauge invariant. A similar problem appears in quantum field theory. There we can consider field operators that transform under a gauge transformation in a similiar way as our matter fields, i.e.,

$$\psi(x) \to \exp\left[ie\Lambda(x)\right]\,\psi(x)\ , \tag{4.123}$$

for a gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \ . \tag{4.124}$$

However, when considering a product of two field operators that should be point splitted in order to avoid ultraviolet divergencies in quantum electrodynamics, the following combination is used to ensure gauge invariance:

$$\bar{\psi}(x+\xi) \exp\left[ie \int_x^{x+\xi} A_\mu \mathrm{d}x^\mu \right] \psi(x) \ . \tag{4.125}$$

The analogue on a lattice is

$$\theta\left(\mathbf{0}, \boldsymbol{R}\right) \quad = \quad \mathrm{e}^{-i\theta(\mathbf{0})} \exp\left[i \sum_C \theta_\ell(\boldsymbol{n}) \right] \mathrm{e}^{i\theta(\boldsymbol{R})} \ , \tag{4.126}$$

where C is a lattice contour that goes from $\mathbf{0}$ to $\boldsymbol{R}$. Taking into account the commutation relation (4.92) we have

$$\left[\mathrm{e}^{\pm i\theta_k(\boldsymbol{n}')}, L_i(\boldsymbol{n}) \right] = \mp\, \delta_{ik}\, \delta_{\boldsymbol{n}\boldsymbol{n}'}\, \mathrm{e}^{\pm i\theta_k(\boldsymbol{n}')} \ . \tag{4.127}$$

Applying the equation to an eigenstate of $L_i \mid n\rangle$, we see that

$$L_i \mathrm{e}^{\pm i\theta_k(\boldsymbol{n})} \mid n\rangle \quad = \quad (n \pm 1) \mathrm{e}^{\pm i\theta_k(\boldsymbol{n})} \mid n\rangle \ . \tag{4.128}$$

That is, each operator on the contour C raises the eigenvalue of L_i by one unit, so that the electric flux passing through that link has been increased by g. This shows that the construction chosen leads to a charge that is a source of g units of electric flux, in agreement with our expectation from Gauss' law.

We can ask next, which contour C will lead to a state of minimum energy. The answer will depend on the value of the coupling constant g. Let us first assume that $g \gg 1$. Looking back at (4.100), we see that the electric term in H dominates. In zeroth order, we can consider

$$aH_0 = \frac{1}{2} \sum_{\boldsymbol{n},k} L_k^2(\boldsymbol{n}) \ , \tag{4.129}$$

where the spectrum is given by the eigenvalues of L_i. The ground-state is given by the eigenvalue zero, that is by the absence of charges. If we now consider the state

$$\theta\left(\mathbf{0}, \boldsymbol{R}\right) \mid 0\rangle \ , \tag{4.130}$$

since on each link on the contour the value of L_i^2 is raised by unity, the lowest energy is obtained by the shortest countour between $\mathbf{0}$ and $\boldsymbol{R}$. This implies that the flux line is a straight line between the charges, and the energy of the state grows linearly with R, giving confinement by a linear potential, as obtained in our previous analysis. As g is decreased, the magnetic part of the Hamiltonian becomes more important. From a perturbative point of view we can expect that the flux lines will meander more, leading to complicated contours. At some moment we expect confinement to cease, since at weak coupling the Coulomb law should prevail, as we have seen in our weak coupling analysis, such that charges may deconfine.