5 Non-Abelian lattice gauge theories

One of the goals of lattice gauge theory is to allow for non-perturbative approaches to the description of elementary particles. As it is well known by now, those gauge theories are characterized by non-Abelian groups. To make the discussion simpler, and since it could be also potentially of interest for other fields of physics, like condensed matter, we will carry out the discussion based on the group SU(2).

5.1 General formulation of the SU(2) theory

After having discussed lattice gauge theories for the groups $\mathbb{Z}_2$ and U(1), a pattern may have merged to discusse how to generalize our considerations up to now to non-Abelian local symmetries. Non-Abelian local gauge theories are known under the name of Yang-Mills theories, since these authors were the first to propose how to generalize gauge invariance to non-Abelian groups [14].

As in previous cases we consider a hypercubic lattice in d-dimensional Euclidean space-time. In a similar fashion as we discussed it in the Abelian case, we imagine a local frame of reference at each site. In the case of SU(2) this means that such a reference frame gives the direction of three orthogonal axis corresponding to the three generators of SU(2), i.e. for the three Pauli matrices, τ_x, τ_y, and τ_z. We allow for the freedom of orienting the reference frame arbitrarily from site to site, and the internal states of a particle (color) have to be described according to the given reference frame on each site. The aim will be to construct an action that is invariant to changes in the orientation of the local color frames of reference. This will lead to the appearence of a colorful gauge field.

Let us consider two sites nearest neighbor to each other, one at site n and the other at site $n + \mu$. Then, the relative orientation of the two frames of reference corresponds to a rotation matrix living on the link between the two sites.

$$U_\mu(n) \quad = \quad \mathrm{e}^{iB_\mu(n)} \; , \tag{5.1}$$

where

$$B_\mu \quad = \quad \frac{1}{2} a \tau_i A^i_\mu(n) \; , \tag{5.2}$$

with $i = 1, 2, 3$. Notice that we included a factor a, as we also did in the Abelian case, that will allow us to go to the continuum as in that case. Also as in the Abelian case, we specify

$$U_{-\mu}(n + \mu) \quad = \quad U^{-1}_\mu(n) \; . \tag{5.3}$$

Once we specified how the gauge fields enter on the links, we have to consider a gauge transformation. It consists of local rotations of the reference frames. The action of such a rotation will be specified in general by local rotations on each site.

Then, the SU(2) matrix $U_\mu(n)$ will be affected by a roation on site n and on site $n+\mu$ as follows:

$$\begin{aligned} [U_\mu(n)]_{ij} \quad \rightarrow \quad & \sum_{kl} \left\{ \exp\left[-i\frac{1}{2}\tau_m \chi^m(n)\right]\right\}_{ik} \\ & \times \left\{ \exp\left[i\frac{1}{2}\tau_p \chi^p(n+\mu)\right]\right\}_{j\ell} [U_\mu(n)]_{k\ell} \, , \end{aligned} \tag{5.4}$$

where the SU(2) matrix

$$\exp\left[-i\frac{1}{2}\tau_m \chi^m(n)\right] \tag{5.5}$$

gives the local rotation at site n.

As in the previous cases, once we define the gauge transformations, the next step is to set up an action that is locally gauge invariant. In a similar way as in the Ising and Abelian lattice gauge theories, we take

$$S = -\frac{1}{2g^2} \sum_{n,\mu\nu} \{\mathrm{Tr}\,[U_\mu(n)U_\nu(n+\mu)U_{-\mu}(n+\mu+\nu)U_{-\nu}(n+\nu)] + \mathrm{h.c.}\} \; . \tag{5.6}$$

It is interesting to see whether the continuum limit of the action above leads, as in the Abelian case, to a known field-theory. For that purpose we consider an expansion in gradients of the gauge fields.

$$\begin{aligned} B_\nu(n+\mu) &\simeq B_\nu(n) + a\partial_\mu B_\nu(n) \; , \\ B_{-\mu}(n+\mu+\nu) &= -B_\mu(n+\nu) \simeq -\left[B_\mu(n) + a\partial_\nu B_\mu(n)\right] \; , \\ B_{-\nu}(n+\nu) &= -B_\nu(n) \; . \end{aligned} \tag{5.7}$$

Then, for the product of the matrices U around a plaquette we have

$$\begin{aligned} & U_\mu(n)U_\nu(n+\mu)U_{-\mu}(n+\mu+\nu)U_{-\nu}(n+\nu) \\ & \quad \simeq \exp\left(iB_\mu\right) \exp\left[i\left(B_\nu + a\partial_\mu B_\nu\right)\right] \exp\left[-i\left(B_\mu + a\partial_\nu B_\mu\right)\right] \exp\left(-iB_\nu\right) \; . \end{aligned} \tag{5.8}$$

Using the Baker-Hausdorff formula

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+(1/2)[A,B]+\cdots} \tag{5.9}$$

it is possible to join some of the terms in the exponents. In doing so, we will keep only contributions up to $\mathcal{O}\left(a^2\right)$, where we take into account the factor a entering in (5.2). Then, we have

$$\begin{aligned} (5.8) \quad \simeq \quad & \exp\left\{i\left(B_\mu + B_\nu + a\partial_\mu B_\nu\right) - \frac{1}{2}\left[B_\mu, B_\nu\right]\right\} \\ & \times \exp\left\{-i\left(B_\mu + B_\nu + a\partial_\nu B_\mu\right) - \frac{1}{2}\left[B_\mu, B_\nu\right]\right\} \\ \simeq \quad & \exp\left\{ia\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right) - \left[B_\mu, B_\nu\right]\right\} = \exp\left(ia^2 g \mathcal{F}_{\mu\nu}\right) \; , \end{aligned} \tag{5.10}$$

where we have defined

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right] , \tag{5.11}$$

with

$$A_\mu = \frac{1}{2}\tau_i A^i_\mu . \tag{5.12}$$

Equation (5.11) is the field-strength found by Yang and Mills for non Abelian gauge theories. The first two terms have the same form as in an Abelian gauge theory. The last term is a consequence of local gauge invariance and the non-Abelian properties of the gauge fields. In the continuum limit we can expand (5.8) further, such that

$$\begin{aligned} \mathrm{Tr}\exp\left(ia^2 g\mathcal{F}_{\mu\nu}\right) &\simeq \mathrm{Tr}\left[1 + ia^2 g\mathcal{F}_{\mu\nu} - \frac{1}{2}a^4 g^2 \mathcal{F}^2_{\mu\nu}\right] \\ &= \mathrm{Tr}\left[1 - \frac{1}{2}a^4 g^2 \mathcal{F}^2_{\mu\nu}\right] , \end{aligned} \tag{5.13}$$

where the linear term in $\mathcal{F}_{\mu\nu}$ vanishes when taking the trace. Due to the algebra of the Pauli matrices,

$$\left[\tau_i, \tau_j\right] = 2i\varepsilon_{ijk}\tau_k , \tag{5.14}$$

we have

$$\mathrm{Tr}\mathcal{F}^2_{\mu\nu} = \frac{1}{2}\left(\partial_\mu A^k_\nu - \partial_\nu A^k_\mu - g\varepsilon_{kij}A^i_\mu A^j_\nu\right)^2 , \tag{5.15}$$

such that the action becomes now

$$\begin{aligned} S &\simeq \frac{1}{2g^2}\int \frac{\mathrm{d}^4x}{a^4} a^4 g^2 \frac{1}{2}\left(\partial_\mu A^k_\nu - \partial_\nu A^k_\mu - g\varepsilon_{kij}A^i_\mu A^j_\nu\right)^2 \\ &= \frac{1}{4}\int \mathrm{d}^4x \left(\mathcal{F}^i_{\mu\nu}\right)^2 , \end{aligned} \tag{5.16}$$

where

$$\mathcal{F}^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g\varepsilon^{ijk}A^j_\mu A^k_\nu . \tag{5.17}$$

The action (5.16) is the classical Euclidean action of pure Yang-Mills fields.

5.2 Discussion on non-Abelian theories

In contrast to the previously discussed cases, less is known in non-Abelian gauge theories. We will therefore not be able to present a closed account on the subject. We restrict ourselves to those points, where the kind of arguments presented in the Ising and Abelian case lead to plausible results.

As in previous cases, due to Elitzur's theorem, we consider Wilson's loop such that a gauge invariant correlation function leads to information about the phases. In a similar manner as for the definition of the action, where we took a product of operators U on a plaquette, we take a product on a closed, directed contour C:

$$\prod_C \mathrm{e}^{iB_\mu(n)} \,. \tag{5.18}$$

The expectation value of such an operator is given by

$$\left\langle \mathrm{e}^{iB_\mu(n)} \right\rangle = \frac{Z(C)}{Z} \,, \tag{5.19}$$

where we defined

$$Z(C) = \int \prod_{n,\mu} \mathrm{d}B_\mu(n)\, \mathrm{e}^{-S} \prod_C \mathrm{e}^{iB_\mu(n)} \,, \tag{5.20}$$

with S the action given in (5.6). As before, the strong-coupling expansion is obtained by expanding $\exp(-S)$ in powers of $1/(2g^2)$. Since we are dealing with rotation matrices, the range of variation of $B_\mu(n)$ is $0 < B_\mu(n) \leq 4\pi$. That is, since SU(2) (and the same for any N in SU(N)), is a compact group, the range of $B_\mu(n)$ is bounded. This should be contrasted with our discussion of the Abelian lattice gauge theory, where we decided to take phase variables, such that the variable $agA_\mu(n)$ was periodic. However, this is not the only possible choice in the Abelian case. In any case, by the same arguments we had in the Abelian case, as many powers of $\exp\left[iB_\mu(n)\right]$ have to be present, in order to have a non-zero result. This leads to an area law, such that

$$\left\langle \mathrm{e}^{iB_\mu(n)} \right\rangle \simeq \mathrm{e}^{-F(g^{-2})A} \,, \tag{5.21}$$

where A is the minimal area enclosed by C. Repeating the arguments of the Abelian case, the correlation function corresponds to the force law between static charges. However, in this case, we are dealing with the carges corresponding to the non-Abelian case, namely color. With particles (quarks) in the fundamental representation of SU(2) their fields transform under a local color gauge transformation as

$$\psi_i(n) \rightarrow \left\{ \exp\left[i\frac{1}{2}\tau_\ell \chi^\ell(n) \right] \right\}_{ij} \psi_j(n). \tag{5.22}$$

A product of operators resulting from point splitting has to have the form

$$\sum_{ij} \bar{\psi}_i \left[\mathrm{e}^{iB_\mu(n)}\right]_{ij} \psi_j(n) \,. \tag{5.23}$$

such that the combination is local gauge invariant. This means, that as a quark hops from site n to $n+\mu$, a rotation of its color takes place specified by a gauge field rotation matrix on the link between the sites. For a contour C, the gauge field part of the amplitude is given by $Z(C)$. We can again consider for C a rectangular closed contour, such that the quark-antiquark potential is given by

$$V(R) \;=\; -\lim_{T\to\infty}\frac{1}{T}\ln\left\langle \mathrm{e}^{iB_\mu(n)}\right\rangle , \tag{5.24}$$

such that the area law leads to a linear confining potential

$$V(R) \;\sim\; \mid R\mid . \tag{5.25}$$

Unfortunately, a clear picture for the weak-coupling limit is still lacking (at least a well established one), and hence, we refrain from considering this limit.

An interesting connection of lattice gauge theories appears by considering the two-dimensional SU(2) lattice gauge theory. We choose the temporal gauge, so that $B_0(n) = 0$. Then, the action reduces to

$$\begin{aligned} S &= -\frac{1}{2g^2}\sum_n \mathrm{Tr}\, U_x(n)U_x^{-1}(n+\hat\tau) + \mathrm{h.c.} \\ &= \sum_x\left\{-\frac{1}{2g^2}\sum_\tau \mathrm{Tr}\, U_x(n)U_x^{-1}(n+\hat\tau) + \mathrm{h.c.}\right\} , \end{aligned} \tag{5.26}$$

i.e. the model decomposes in copies of a nearest-neighbor one-dimensional spin model. The matrices U transform according to SU(2)×SU(2), the spin model is an SU(2)×SU(2) spin chain. The symmetry now is a global one. Furthermore, since SU(2)×SU(2) $\simeq$ O(4), we expect this model to be the O(4) Heisenberg model. In fact, this can be seen by parametrizing $U_x(n)$ in terms of Pauli matrices:

$$U_x(n) \;=\; \sigma(n) + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}(n) , \tag{5.27}$$

where $(\sigma, \boldsymbol{\pi})$ are four real fields. Since $U_x(n)$ is unitary, the real fields have to satisfy the condition

$$\sigma^2(n) + \boldsymbol{\pi}^2 \;=\; 1 . \tag{5.28}$$

We can then consider these fields as the four components of a unit vector in four dimensions:

$$\boldsymbol{S}(n) \;=\; \begin{pmatrix}\sigma\\ \pi_1\\ \pi_2\\ \pi_3\end{pmatrix} , \quad \mid \boldsymbol{S}(n)\mid^2 = 1 . \tag{5.29}$$

Furthermore, we have

$$\begin{aligned} \mathrm{Tr}\, U_x(n)U_x^{-1}(n+\hat\tau) &= 2\left[\sigma(n)\,\sigma(n+\hat\tau) + \boldsymbol{\pi}(n)\cdot\boldsymbol{\pi}(n+\hat\tau)\right] \\ &= 2\,\boldsymbol{S}(n)\cdot\boldsymbol{S}(n+\hat\tau) , \end{aligned} \tag{5.30}$$

such that the action becomes

$$S = \sum_x \left\{ -\frac{2}{g^2} \sum_\tau \boldsymbol{S}(n) \cdot \boldsymbol{S}(n+\hat{\tau}) \right\} , \tag{5.31}$$

as anticipated.

The one-dimensional spin system is disordered at any finite temperature by virtue of the Mermin-Wagner theorem. As we have already seen in the case of the two-dimensional Ising gauge theory, that corresponded to the one-dimensional Ising model, again, the area law, and hence, confinement, can be identified with an exponential decay of the correlation function. There are further analogies to what is known about non-Abelian gauge theories. The renormalization group analysis of the O(n) Heisenberg model in field theoretic terms (the non-linear σ-model) shows that it also displays asymptotic freedom and dynamical mass generation. Due to these facts, we review shortly the renormalization group analysis for O(n) Heisenberg models in the next section.

5.3 A short overview on the renormalization group analysis of O(N) spin systems in two dimensions

Let us consider a Heisenberg spin-model with the action

$$S_H = -\frac{1}{2g} \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j , \tag{5.32}$$

where $\langle i,j \rangle$ means that the interaction is only for j nearest neighbor of i on a square lattice. We consider here the two dimensional case with the expectation that a SU(2) lattice gauge theory in 4 dimensions shares some properties with a spin system in two dimensions [15]. These properties will turn out to be asymptotic freedom and dynamical genration of mass. We will concentrate on the weak coupling limit (or equivalently the low temperature limit), where we can expect that the system is almost ordered, that is, the spins are correlated over distances $\xi \gg a$, where a is the lattice constant. The spins have N components, and fulfill the constraint

$$|\boldsymbol{S}_i| = 1 . \tag{5.33}$$

Such a constraint will allow only for transverse fluctuations. Due to Mermin-Wagner's theorem, the systems will show order only at $g = 0$. For small g only fluctuations around the ordered state will be important, and hence, we concentrate on them.

The partition function is

$$Z = \int \prod_{i=1}^{M} \mathrm{d}^N \boldsymbol{S}_i \, \delta\left(\boldsymbol{S}_i^2 - 1\right) \mathrm{e}^{-S_H} , \tag{5.34}$$

where M is the number of lattice sites. In the case that ξ, the spin-spin correlation length, is very large, only small deviations of an ordered state will be present, so that a gradient expansion is appropriate.

$$\boldsymbol{S}_j \simeq \boldsymbol{S}_i + a^\mu \frac{\partial}{\partial x^\mu} \boldsymbol{S}_i + \frac{1}{2} a^\mu a^\nu \frac{\partial^2}{\partial x^\mu \partial x^\nu} \boldsymbol{S}_i + \cdots \tag{5.35}$$

Due to the constraint (5.33), we have

$$\boldsymbol{S}_i \cdot \frac{\partial}{\partial x^\mu} \boldsymbol{S}_i = 0 , \tag{5.36}$$

and

$$\frac{\partial}{\partial x^\mu} \left(\boldsymbol{S}_i \cdot \frac{\partial}{\partial x^\nu} \boldsymbol{S}_i \right) = \frac{\partial}{\partial x^\mu} \boldsymbol{S}_i \cdot \frac{\partial}{\partial x^\nu} \boldsymbol{S}_i + \boldsymbol{S}_i \cdot \frac{\partial^2}{\partial x^\mu \partial x^\nu} \boldsymbol{S}_i = 0 , \tag{5.37}$$

such that we finally obtain the action of the O(N) non-linear σ-model.

$$S_H \rightarrow S_{NL\sigma M} = \frac{1}{2g} \int \mathrm{d}^2 x \left(\partial_\mu \boldsymbol{S} \right)^2 . \tag{5.38}$$

Next we discuss the treatment of the constraint (5.33) in the path integral. We define first

$$\begin{aligned} \pi^\alpha(x) &\equiv S_i^\alpha , \qquad \alpha = 1, \ldots, N-1 , \\ \sigma(x) &\equiv S_i^N = \sqrt{1 - \boldsymbol{\pi}^2(x)} . \end{aligned} \tag{5.39}$$

In such a way, the action becomes

$$S_{NL\sigma M} = \frac{1}{2g} \int \mathrm{d}^2 x \left[\left(\partial_\mu \boldsymbol{\pi} \right)^2 + \left(\partial_\mu \sqrt{1 - \boldsymbol{\pi}^2} \right)^2 \right] . \tag{5.40}$$

For the measure of the path integral we have moreover,

$$\int \mathrm{d}S_i^N \, \delta \left[\left(S_i^N \right)^2 - \left(1 - \boldsymbol{\pi}_i^2 \right) \right] = \frac{1}{2} \int \frac{\mathrm{d}x_i}{\sqrt{x_i}} \delta \left[x_i - \left(1 - \boldsymbol{\pi}_i^2 \right) \right] , \tag{5.41}$$

that leads to a new measure

$$\begin{aligned} \prod_{i=1}^M \frac{\mathrm{d}^{N-1} \pi_i}{\sqrt{1 - \boldsymbol{\pi}^2}} &= \mathcal{D}\boldsymbol{\pi} \, \exp \left[-\frac{1}{2} \sum_i \ln \left(1 - \boldsymbol{\pi}^2 \right) \right] \\ &= \mathcal{D}\boldsymbol{\pi} \, \exp \left[-\frac{1}{2} a^{-d} \int \mathrm{d}^d x \ln \left(1 - \boldsymbol{\pi}^2 \right) \right] . \end{aligned} \tag{5.42}$$

The partition function has now the form

$$\begin{aligned} Z = \int \mathcal{D}\boldsymbol{\pi} \, \exp \Bigg\{ &- \frac{1}{2g} \int \mathrm{d}^2 x \left[\left(\partial_\mu \boldsymbol{\pi} \right)^2 + \left(\partial_\mu \sqrt{1 - \boldsymbol{\pi}^2} \right)^2 \right] \\ &- \frac{1}{2} a^{-2} \int \mathrm{d}^2 x \ln \left(1 - \boldsymbol{\pi}^2 \right) \Bigg\} . \end{aligned} \tag{5.43}$$

Here we see that the rotational invariance in $N-1$ directions is implemented lineraly, while rotations on a plane containing σ are implemented in a non-linear fashion. In this way, the constraint is eliminated at the expense of introducing non-linear expressions. However, since we can assume that the deviations from the ordered state are small, we can perform an expansion in powers of $\boldsymbol{\pi}$.

It should be noticed, that in two dimensions, both the fields and the coupling constant are dimensionless. Hence, by performing a change of scale, the coupling will not change, i.e. it is *scale invariant*. The dimension, where the coupling constant is dimensionless is called the *critical dimension*. At that dimension the theory is called *renormalizable*, since the divergences appearing in perturbation theory can be render finite with a finite number of *renormalization constants*.

Without being able to enter into a detailed discussion of perturbation theory, for which we refer to the well established literature [16, 17] or past lectures[1], and at the risk of remaining on a too formal level, we delineate in the following the steps to perform a renormalization group analysis of the non-linear σ-model along field-theoretic lines.

From the form of the action (5.38), we see that an expansion in the weak coupling limit corresponds to an expansion in fluctuations around the saddle point, that becomes exact for $g \to 0$. Such an expansion is called a *loop expansion*. In order to see what is meant by that, we start by realising that an expansion in powers of g is equivalent to an expansion in powers of the field $\boldsymbol{\pi}$ in (5.43). This can be seen by changing to new variables

$$\tilde{\boldsymbol{\pi}} = \frac{\boldsymbol{\pi}}{\sqrt{g}} \,, \tag{5.44}$$

However, due to the term coming from the measure, a negative mass $\sim a^{-2}$ would be present. This is actually not a fundamental problem, since it is possible to show that this mass will be cancelled order by order in perturbation theory [16]. Nevertheless, it would be desirable to dispose of it from the beginning. This can be achieved within *dimensional regularization* [16, 17, 18], a way of regularizing integrals appearing in the computation of Feynman diagrams by performing an analytic continuation of the number of dimensions d from a natural number to the complex plane. Specifically, we consider here the mass

$$a^{-2} = \frac{M}{a^2 M} = \frac{1}{(2\pi)^2} \sum_{\boldsymbol{q}} \left(\frac{2\pi}{L} \right)^2 = \int_0^\Lambda \frac{\mathrm{d}^2 q}{(2\pi)^2} \,. \tag{5.45}$$

such an integral can be viewed as a special case of

$$\int_0^\Lambda \frac{\mathrm{d}^d q}{(2\pi)^d}\, q^\nu \;=\; \int \mathrm{d}\Omega_d \int_0^\Lambda \mathrm{d}q\, q^{d-1+\nu} \,, \tag{5.46}$$

where we introduced an ultraviolet cutoff Λ. Since the first integral corresponds just to the surface of the unit sphere in d dimensions, we can disregard it for the present

[1] http://www.theo3.physik.uni-stuttgart.de/lehre/ws05/PhaseT

discussion. We concentrate then on

$$\begin{aligned} I &= \int_0^\Lambda \mathrm{d}q\, q^{d-1+\nu} \\ &= q q^{d+\nu-1}\Big|_0^\Lambda - (d+\nu-1)\int_0^\Lambda \mathrm{d}q\, q^{d-1+\nu} \, . \end{aligned} \tag{5.47}$$

Formally, after the integration by parts, we can disregard the surface term for $\nu = -d$. To disregard it for other values of ν means that the integral itself should vanish. This is the way such integrals are treated in dimensional regularization. This means that in this frame, the integral (5.45) is discarded.

Once we disposed of the measure in dimensional regularization, we are left with the two other terms in the argument of the exponential in (5.43). the first one can be written as follows

$$\frac{1}{2g}\int \mathrm{d}^2x\,(\partial_\mu \boldsymbol{\pi})^2 = \frac{1}{2}\int \mathrm{d}^2x\,\mathrm{d}^2x' \pi^a(x) G_0^{-1}(x,x')\pi^a(x') \, , \tag{5.48}$$

where the inverse of the free two-point propagator is

$$G_0^{-1}(x,x') = \frac{1}{g}\triangle \, , \tag{5.49}$$

with $\triangle$ the Laplace operator, in this case in two dimensions. Fourier transforming this expression and inverting it, it is clear that an infrared divergence appears:

$$G_0^{(\alpha)}(\boldsymbol{p}) = \frac{g}{\boldsymbol{p}^2} \, . \tag{5.50}$$

This means that we have to regularize the theory in the infrared by inserting a mass. In a spin system, such a mass is introduced by a magnetic field. By inserting a magnetic field H in the action, acting in the σ-direction, we have an additional term in the action of the form

$$\begin{aligned} -H\int \mathrm{d}^2x\,\sigma(x) &= -H\int \mathrm{d}^2x\,\sqrt{1-\boldsymbol{\pi}^2} \\ &= -H\int \mathrm{d}^2x\left[1 - \frac{1}{2}\boldsymbol{\pi}^2 - \frac{1}{8}\left(\boldsymbol{\pi}^2\right)^2 \cdots \,\right] \end{aligned} \tag{5.51}$$

Then, the free propagator becomes

$$G_0^{(\alpha)}(\boldsymbol{p}) = \frac{g}{\boldsymbol{p}^2 + H} \, . \tag{5.52}$$

The loop expansion can be shown to correspond to a perturbative treatment of so-called vertex-functions [16, 17]. Here we will not enter into the general discussion of them but just schematically show the steps leading to the renormalization group equations. Let us state here that in the case of the non-linear σ-model, only the two-point vertex function needs to be considered. It corresponds to the inverse of the two-point propagator, so that in lowest order it is given by

$$\Gamma_{\alpha}^{(2)} = \frac{1}{g}\left(\boldsymbol{p}^2 + H\right) . \tag{5.53}$$

In order to have higher corrections, the methods of perturbation theory with Feynman diagrams have to be used. They lead to an expansion where the contributions are obtained in form of integrals. Each integral corresponds to one loop. The results are better seen in the frame of an ϵ-*expansion*, where $\epsilon = d - 2$. Up to *one loop*, the result is

$$\Gamma_{\alpha}^{(2)} = \frac{1}{g}\left(\boldsymbol{p}^2 + H\right) - \left[\frac{1}{2}(N-1)H + \boldsymbol{p}^2\right]\frac{1}{\epsilon}H^{\epsilon/2} + \mathcal{O}\left(\epsilon, g\right) . \tag{5.54}$$

The divergences of the theory are now concentrated in the pole in ϵ. In order to render the theory finite, we introduce a renormalized coupling

$$g = \kappa^{-\epsilon} Z_g g_R , \tag{5.55}$$

where κ denotes a scale (of momentum or energy) at which the renormalization is performed. It should be there, since if we depart from $d = 2$, the coupling constant becomes dimensionfull and scales as a^{d-2}, or in momentum space as $\kappa^{-\epsilon}$. Z_g is a renormalization constant that will be used to render the renormalized theory finite. Also the fields have to be renormalized (*wave function renormalization*) by introducing a corresponding renormalization constant:

$$\pi^{\alpha} = Z_{\pi}^{1/2}\pi_R^{\alpha} . \tag{5.56}$$

The renormalization of the fields implies also a renormalization of the magnetic field, since σ has the same dimensions as the field π^{α}. Here we set

$$H = Z_g Z_{\pi}^{-1/2} h . \tag{5.57}$$

Once the coupling and fields are renormalized, also the vertex functions are renormalized. For the two-point vertex function we have

$$\Gamma_{\pi R}^{(2)}\left(\boldsymbol{p}, g_R, h, \kappa\right) = Z_{\pi}\Gamma_{\pi}^{(2)}\left(\boldsymbol{p}, g, H, \Lambda\right) . \tag{5.58}$$

Since we considered only contributions up to $\mathcal{O}(g)$, the renormalization constants can be only given up to the same order.

$$\begin{aligned} Z_g &= 1 + a g_R + \mathcal{O}\left(g_R^2\right) , \\ Z_{\pi} &= 1 + b g_R + \mathcal{O}\left(g_R^2\right) . \end{aligned} \tag{5.59}$$

The constants a and b have to be determined such that the pole in ϵ is cancelled, such that the expressions become finite. The result is

$$a = \frac{N-2}{\epsilon} , \qquad b = \frac{N-1}{\epsilon} . \tag{5.60}$$

An important element in the renormalizatiopn group analysis is played by the β-*function*. It can be obtained by formulating the *Callan-Symanzik equations* and shows, how the coupling constant renormalizes upon a change of scale. For the renormalized coupling constant it is given by

$$\beta(g_R,\kappa) = \kappa \left.\frac{\partial g_R}{\partial \kappa}\right|_g . \qquad (5.61)$$

Inserting the results obtained, we arrive at

$$\beta(g_R,\kappa) = \epsilon g_R - (N-2) g_R^2 + \mathcal{O}\left(g_R^3\right) . \qquad (5.62)$$

Although we are primarily interested in the two dimensional case, i.e. $\epsilon = 0$, let us discuss the result obtained also for $\epsilon > 0$.

In two dimensions, the β-funtion is negative. This means that as we increase

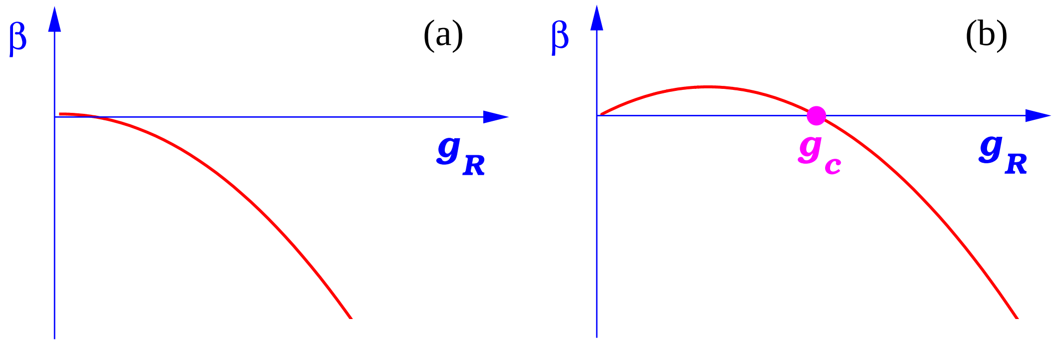

Figure 10: Beta functions for (a) $\epsilon = 0$ and (b) for $\epsilon > 0$.

the energy scale, the coupling renormalizes towards zero. Therefore, in the UV limit the theory displays *asymptotic freedom*. This is schematically shown in Fig. 10 (a). On the contrary, in the IR limit, i.e. for decreasing energy scales, the theory renormalizes towards strong coupling. Extrapolating to lattice gauge theories, this would mean, that confinement should be expected as we lower the energy scales.

For dimensions higher than 2, i.e. for $\epsilon > 0$, the β-function has a zero at a finite value of the coupling constant. This happens at

$$g_c = \frac{\epsilon}{N-2} . \qquad (5.63)$$

At this point, the coupling constant is scale invariant. In the IR limit and for $g < g_c$, the coupling constant scales towards zero, while for $g > g_c$, it scales towards larger values. Therefore, g_c corresponds to an *unstable fixed point* that devides the ordered phase ($g < g_c$) from the disordered one, and hence, it is the *critical coupling*. We can see there, that the whole treatment breaks down for $N = 2$ and $\epsilon = 0$, where in fact, we expect a rather different behavior since there is a Kosterlitz-Thouless transition.

Finally, let us mention that since in two dimensions the non-linear σ-model scales to strong coupling, the systems is in the disordered phase, and a finite correlation length should be present. This means for the excitations of the model, that they are massive, and hence, even in the limit $H \to 0$, a mass is generated by fluctuations. It can be calculated in the frame of the present perturbative expansion [16], but this would go beyond the scope of the present lectures.

5.4 Coupling to matter fields in a lattice gauge theory

In the last part of these lectures, we discuss the coupling of matter fields in a lattice gauge theory. As will be seen below, the introduction of dynamical matter fields may change qualitatively the phase diagrams that we obtained for pure gauge theories, where confinement was discussed in terms of static charges. The main change will come from the possibility of creating quark-antiquark pairs out of the vacuum, when separating two static charges a distance R in the confining phase, since this may turn to be energetically favorable. In such a way, the dynamical quarks may screen the long-range confining potential, rendering it short ranged. Although the coupling to a scalar matter field was studied for $\mathbb{Z}_2$, U(1), and SU(N) theories [19], we will restrict ourselves to the Ising case, as an illustration of the main ideas.

The action of pure Ising gauge fields was already given in (3.1). We add to it a gauge invariant part for *Ising matter fields* of the form

$$\sigma(n) \;=\; \pm 1 \,, \tag{5.64}$$

that are localized at the sites of the lattice. The gauge fields are called here

$$U_\mu(n) \;=\; \pm 1 \,, \tag{5.65}$$

in accordance with the denomination introduced in the present chapter. The gauge-invariant action is given now by

$$\begin{aligned} S \;=\; & \beta \sum_{n,\mu} \sigma(n)\, U_\mu(n)\, \sigma(n+\mu) \\ & + K \sum_{n,\mu\nu} U_\mu(n) U_\nu(n+\mu) U_\mu(n+\nu) U_\nu(n) \,. \end{aligned} \tag{5.66}$$

While for the Ising model it is not direct how to pass to continuous fields, it should be clear in the U(1) case, that the first term corresponds to the covariant derivative for matter fields, if we identify

$$U_\mu(n) \;\to\; \mathrm{e}^{i\theta_\mu(n)} \,, \tag{5.67}$$

and we go over to the continuum using (4.15). A local gauge transformation consists in flipping the matter field at n and the gauge fields at all the links connected to that site.

A first insight into the phase diagram of the model can be achieved by considering two limiting cases, $K = \infty$, and $\beta = 0$ [19].

1. $K = \infty$. In this situation, the gauge fields are frozen, such that the model reduces to the conventional Ising model. For $d \geq 2$ the Ising model has two phases with $\langle \sigma(n) \rangle \neq 0$ for $\beta > \beta_c$ and $\langle \sigma(n) \rangle = 0$ otherwise. The fact that the expectation value of the scalar field acquires a non-vanishing value can be viewed as a Higgs mechanism taking place, such that $\sigma(n)$ is referred to as Higgs-field [19].

2. $\beta = 0$. In this case we come back to the pure gauge Ising model, as discussed in Chapter 3. There we have seen that there are also two phases, one characterized by the area law, and hence confining, and the other by a perimeter law, where the static charges are free.

As a next step, we can consider how these critical points extend into the interior of the phase diagram. We first consider β small such that an expansion in β can be performed. Since this is considered to be a small perturbation, we expect that the fields σ will just renormalize K. Hence, by tracing out the field σ, an effective action for the pure gauge fields should result.

$$\exp\left\{S_{\text{eff}}\left[U_\mu(n)\right]\right\} = \sum_{\{\sigma\}} \exp\left[\beta \sum \sigma\, U\, \sigma + K \sum UUUU\right] . \tag{5.68}$$

As we already did in the case of the conventional Ising model, we use the identity

$$\exp\left(\beta \sigma\, U\, \sigma\right) = \cosh\beta\left(1 + \sigma U \sigma \tanh\beta\right) . \tag{5.69}$$

The sum over the configurations of σ can only be non-zero when these fields are squared. The first non-vanishing contribution in lowest order in $\tanh\beta$ is obtained on a plaquette, i.e. in $\mathcal{O}\left(\tanh^4\beta\right)$:

$$\begin{aligned}\exp\left\{S_{\text{eff}}\left[U_\mu(n)\right]\right\} &= \mathrm{e}^{K\sum UUUU}\left[1 + \tanh^4\beta \prod UUUU + \cdots\right] \\ &\simeq \exp\left[\left(K + \tanh^4\beta\right)\sum UUUU\right] .\end{aligned} \tag{5.70}$$

Hence, we arrive at an action of a pure gauge model with an effective coupling

$$K_{\text{eff}} = K + \tanh^4\beta . \tag{5.71}$$

Calling the critical value of the pure Ising gauge theory K_c^0, the critical value for the gauge theory with Higgs fields is given by

$$K_c = K_c^0 - \tanh^4\beta , \tag{5.72}$$

such that the transition between a confining and a free phase is shifted towards smaller values of the coupling K as β increases. This transition line is sketched at the bottom of Fig. 11.

In order to discuss the situation at $K \to \infty$, we recall that in Sec. 3.4 we have shown that the three-dimensional Ising gauge theory is dual to the 3D-Ising model.

Hence, the transition point at the bottom of Fig. 11 has to have its dual on the $K = \infty$ line, at least in $d = 3$. In $d = 3$, it also holds that the Ising gauge theory coupled to Higgs fields is self-dual [1, 19]. Therefore, the line starting at the bottom of Fig. 11 has to have its dual starting at the transition point on the $K = \infty$ line.

Since the arguments leading to the extensions of transition lines into the interior

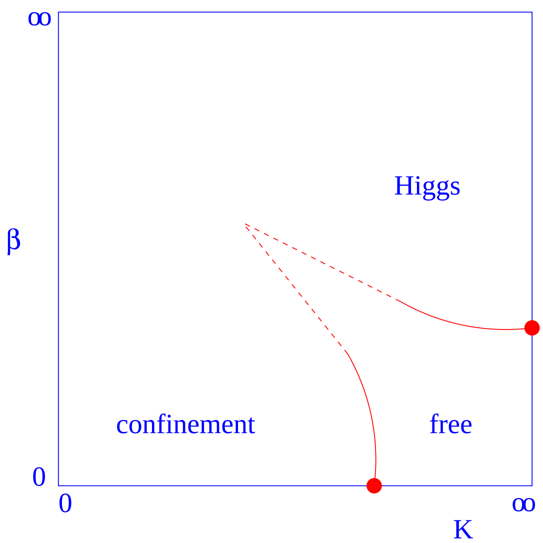

Figure 11: Schematic phase diagram for the $\mathbb{Z}_2$ gauge theory coupled to Higgs-fields.

of the phase diagram are perturbative in nature, dashed lines in the phase diagram indicate how the phase diagram probably is. the fact that the lines meet in hte interior is due to the fact that it can be shown that no singularities appear on a strip at small K or large β. We are not going into a discussion of it, details can be found in the work by Fradkin and Shenker [19]. In the same work, an analysis is performed for a U(1) lattice gauge theory coupled to Higgs-fields, arriving at a similar phase diagram. Also similar conclusions are obtained for non abelian lattice gauge theories with matter fields, albeit the situation with fermions is less clear.

Further readings on lattice gauge theory, that go beyond the introductory material discussed here, can be found in a later review article by Kogut [20].