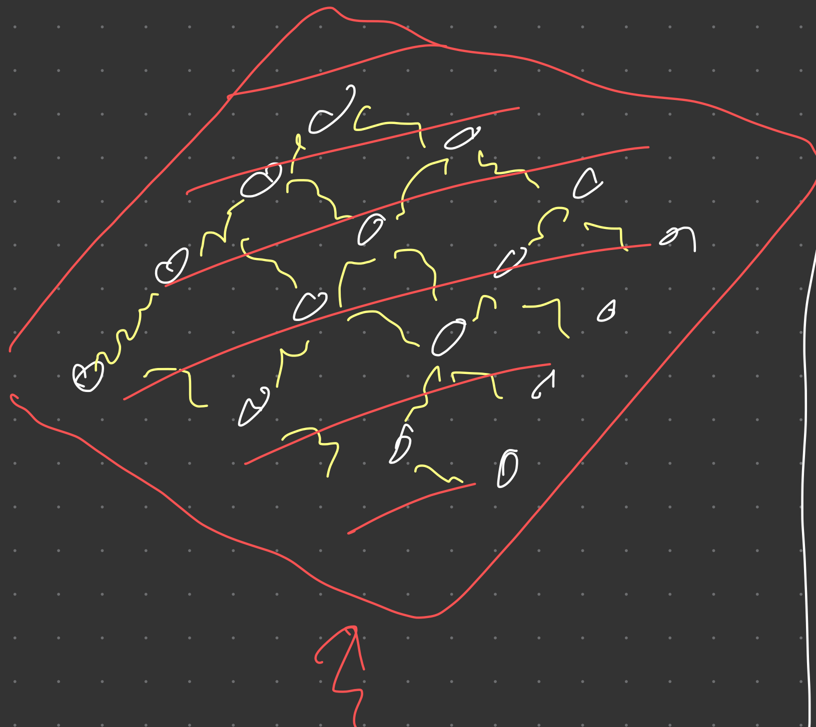
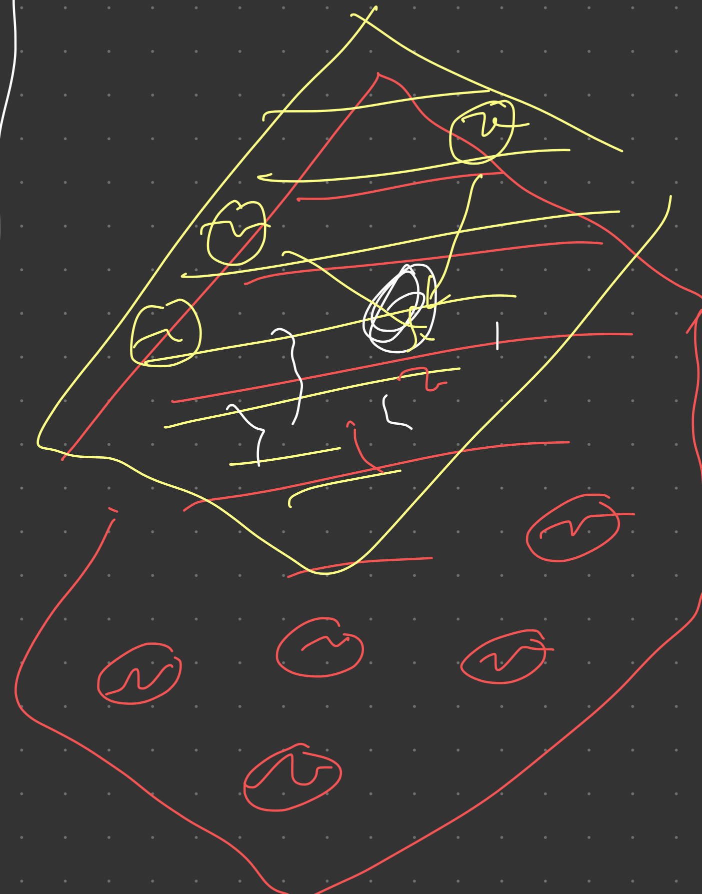




1. Elements of classical field theory

1.1. Lagrangian and Hamiltonian formalism

Recap: Classical mechanics of "points"

1. Degrees of freedom q_i labeled by $i = 1, \dots, N$
2. Lagrangian $L(\{q_i\}, \{\dot{q}_i\}, t) = T - V$
3. Action $S[q] = \int dt\, L(q(t), \dot{q}(t)) \in \mathbb{R}$
4. Hamilton's principle of least action:

$$\frac{\delta S[q]}{\delta q} \overset{!}{=} 0 \iff \delta S = \int dt\, \delta L \overset{!}{=} 0$$

5. Euler-Lagrange equations $(i = 1 \dots N)$:

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0}$$

Analogous: Lagrangian Field Theory

1. One or more fields $\phi(x)$ on spacetime $x \in \mathbb{R}^{1,3} / \mathbb{R}^4$
   with derivatives $\partial_\mu \phi(x)$ where $\partial_0 = \partial_t$ and $\partial_{i=1,2,3} = \partial_{x,y,z}$

$\sum_i$

2. Lagrangian density $\mathcal{L}(\phi, \partial\phi, x) \longrightarrow$ Lagrangian $L = \int d^3x \; \mathcal{L}(\phi, \partial\phi)$

3. Action: $S[\phi] = \int dt\, L = \int dt \int d^3x \; \mathcal{L}(\phi, \partial\phi) = \int d^4x \; \mathcal{L}(\phi, \partial\phi)$

4. Action principle: $0 \overset{!}{=} \delta S[\phi] = \int d^4x \; \delta\mathcal{L}$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\}$$

$\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \right\}$$

Gauss theorem?

$$= \int_\partial dA \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underbrace{\delta\phi}_{=0} + \int d^4x \underbrace{\left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right\}}_{\Longrightarrow\, = 0} \delta\phi$$

5. Euler-Lagrange equations (one for each field):

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0}$$

Recap: Hamiltonian Mechanics

Lagrangian $L(q,\dot{q},t)$ $\xrightarrow{\text{Legendre Transformation}}$ Hamiltonian $H(q,p,t) = p\cdot\dot{q} - L(q,\dot{q},t)$

Conjugate momentum

$$p \equiv \frac{\partial L}{\partial \dot{q}} \Leftrightarrow \dot{q} = \dot{q}(p)$$

Analogous: Hamiltonian Field Theory

1. Let $x = x_i \hat{=} i$ be discrete spatial coordinates

$$\frac{\partial L}{\partial \dot{q}_i} = p_i \hat{=} p(x) = \frac{\partial L}{\partial \dot{\phi}(x)} = \frac{\partial}{\partial \dot{\phi}(x)} \int d^3y \, \mathcal{L}(\phi(y), \dot{\phi}(y)) \sim \sum_y d^3y \underbrace{\frac{\partial}{\partial \dot{\phi}(x)} \mathcal{L}(\phi(y), \dot{\phi}(y))}_{\delta_{x,y} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\big|_{y=x}}$$

$$= \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}}_{\equiv \pi(x)} d^3x$$

$\rightarrow$ Momentum density conjugate to ϕ is $\boxed{\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}}$

2. Hamiltonian:

$$H = \sum_x \overbrace{p(x)}^{\pi(x)\, d^3x} \dot{\phi}(x) - \overbrace{L}^{\sum_x \mathcal{L}(\phi(x), \dot{\phi}(x))\, d^3x}$$

$$= \int d^3x \underbrace{\{ \pi(x) \dot{\phi}(x) - \mathcal{L}(\phi, \dot{\phi}) \}}_{\text{Hamiltonian density } \mathcal{H}(\phi, \pi)}$$

$\dot{\phi} = \dot{\phi}(\pi)$

Example 1.1: Free scalar field

1. Real field $\phi: \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$ with $(\vec{x}, t) \mapsto \phi(\vec{x}, t) = \phi(x)$

2. Lagrangian (density): $\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$

$$(\partial_\mu \phi)^2 \equiv \partial_\mu \phi \, \partial^\mu \phi = (\partial_t \phi)^2 - (\partial_x \phi)^2 - (\partial_y \phi)^2 - (\partial_z \phi)^2 \quad \Big\} \quad \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2$$

3. Interpretation:

potential energy $\sim \phi^2$

$\sim (\partial_t \phi)^2$ kinetic energy

$\sim (\nabla \phi)^2$ "stretching energy"

4. Equation of motion ("field equation"):

$$-m^2 \phi - \partial_\mu (\partial^\mu \phi) = 0 \iff (\partial_\mu \partial^\mu + m^2) \phi = 0$$

(classical Klein-Gordon equation)

5. Conjugate momentum field $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$

6. Hamiltonian density: $\mathcal{H} = \pi \dot{\phi} - \frac{1}{2} \dot{\phi}^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 = \frac{1}{2} \pi^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2$

1.2 Symmetries and Conservation Laws

1. $\exists$ transformations on coordinates and fields:

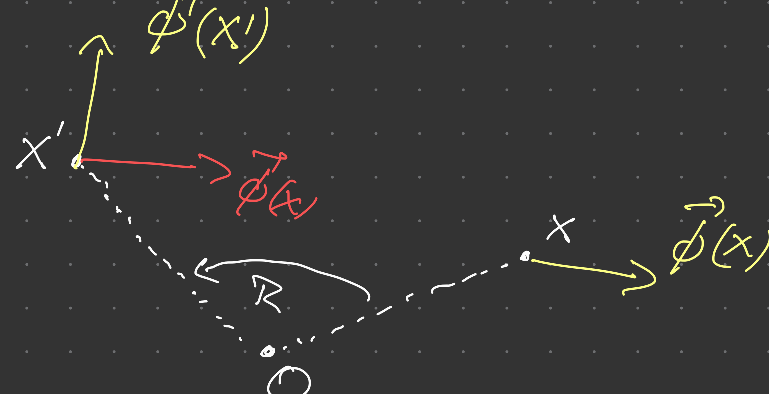
$$x \longrightarrow x' = x'(x) \quad \text{and} \quad \phi(x) \longrightarrow \phi'(x') = \mathcal{F}(\phi(x))$$

Two effects: coordinates and fields transformed

Example 1.2: Rotation of a vector field $\vec{\phi}$

a) $\exists$ 3-component field $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and $R \in SO(3)$ rotation

b) $\vec{x}' = R\vec{x}$ and $\vec{\phi}'(x') = R\vec{\phi}(x) = R\vec{\phi}(R^{-1}x')$



2. Change of the action under $\phi \mapsto \phi'$:

$$S' \equiv S[\phi'] = \int d^dx \, \mathcal{L}(\phi'(x), \partial_\mu \phi'(x))$$

rename $x \to x'$ $\overset{?}{=} \int d^dx' \, \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'))$

definition $\overset{?}{=} \int d^dx' \, \mathcal{L}(\mathcal{F}(\phi(x)), \partial'_\mu \mathcal{F}(\phi(x)))$

$$= \int d^dx \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}\left(\mathcal{F}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F}(\phi(x)) \right)$$

Example 1.3: Translations

defines a scalar field

a) $x' := x + a$, $\phi'(x') := \phi(x) = \phi(x' - a)$

b) $\mathcal{F}$ trivial, $\phi'(x') = \mathcal{F}(\phi(x)) = \phi(x(x'))$, and $\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu$

c) Action:

$$S[\phi'] = \int d^dx \, \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) = \int d^dx \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = S[\phi]$$

$\to$ action is translational invariant