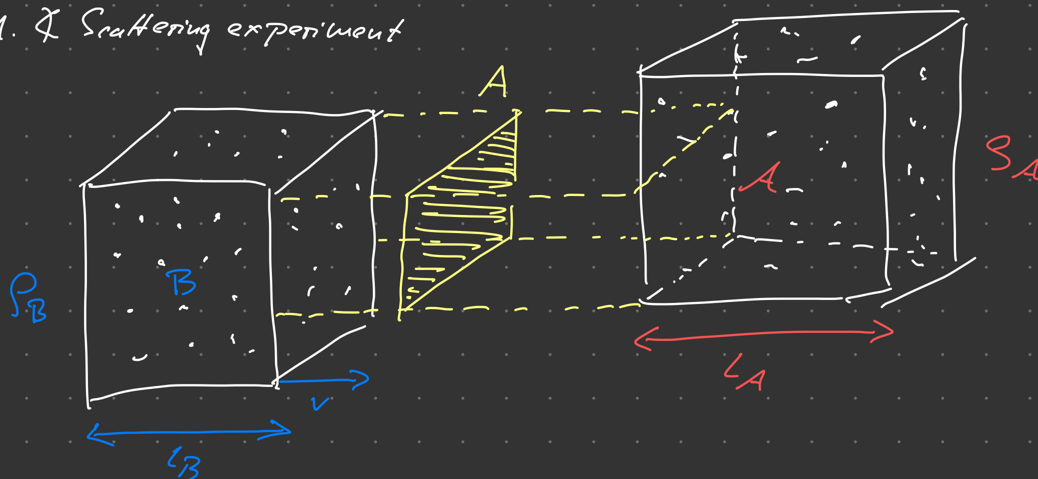
4.5 Cross Sections and the S-Matrix

The Cross Section

1. $\mathbb{X}$ Scattering experiment



2. Cross section

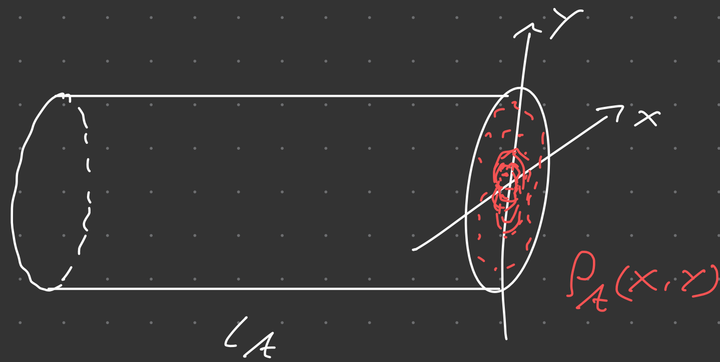
$$\boxed{\sigma_{(X)} \equiv \frac{\#\text{ of scattering events (with outcome } X)}{\rho_A \, l_A \, \rho_B \, l_B \, A}}$$

→ encodes the likelihood of scattering event X

→ intrinsic property of colliding particles

$\Longrightarrow \quad [\sigma] = L^2 = \text{Area}$

3. Real experiments: Densities not homogeneous across beam: $\rho_X \mapsto \rho_X(x,y)$



$$\#\text{ of scattering events } X = \sigma_{(x)} \cdot l_A l_B \underbrace{\int dx\, dy}_{\text{Beam cross section}} \rho_A(x,y)\, \rho_B(x,y)$$

homogeneous beam: $\rho_X = \text{const.}$

$$= \frac{\sigma_{(x)} N_A N_B}{A}$$

N_X: # of particles of type X in interaction volume $l_X \cdot A$

4. Typically there are many outcomes X possible, e.g.

$$e^+e^- \longrightarrow \begin{cases} e^+e^- & \\ \mu^+\mu^- & \text{(muons)} \\ \mu^+\mu^-\gamma & (\gamma\text{: photon}) \\ \cdots & \end{cases}$$

5. Differential cross section

- Scattering outcome X of n final particles with momenta $(\vec{p}_1, \ldots, \vec{p}_n) \in V_p \subseteq \mathbb{R}^{3n}$

V_p: volume of final-state 3-momentum space $\mathbb{R}^{3n}$

$$\sigma_{X|V_p} = \int_{V_p} d^3p_1 \ldots d^3p_n \underbrace{\frac{d\sigma}{d^3p_1 \ldots d^3p_n}}_{\text{differential cross section}}$$

$\rightarrow$ constraint by 4-momentum conservation: $\sum_i p_i = \text{const.}$

Special case: $n=2$

$\rightarrow$ 6 DOF $(\vec{p}_1, \vec{p}_2)$ and 4 constraints

$\rightarrow$ 2 DOF left: scattering direction (ϕ, ϑ) in center-of-mass frame:

$$\frac{d\sigma}{d^3p_1\, d^3p_2} \longrightarrow \frac{d\sigma}{d\Omega}$$

The S-Matrix

1. One-particle wavepacket

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \phi(\vec{k})\, |\vec{k}\rangle \qquad \text{with} \quad \int \frac{d^3k}{(2\pi)^3} |\phi(\vec{k})|^2 = 1 = \langle\phi|\phi\rangle$$

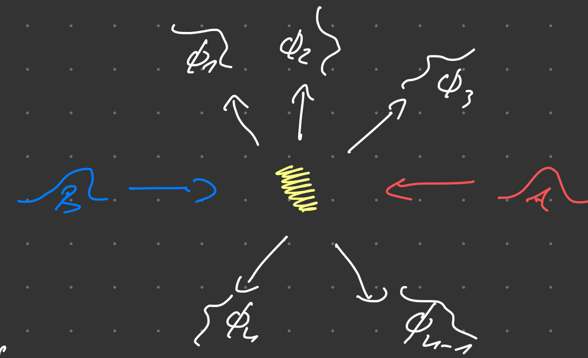
$|\vec{k}\rangle$: one-particle state of interacting theory

for free theory:

$$|\vec{k}\rangle_0 = \sqrt{2E_k}\, a^\dagger_{\vec{k}} |0\rangle$$

2. We want the probability

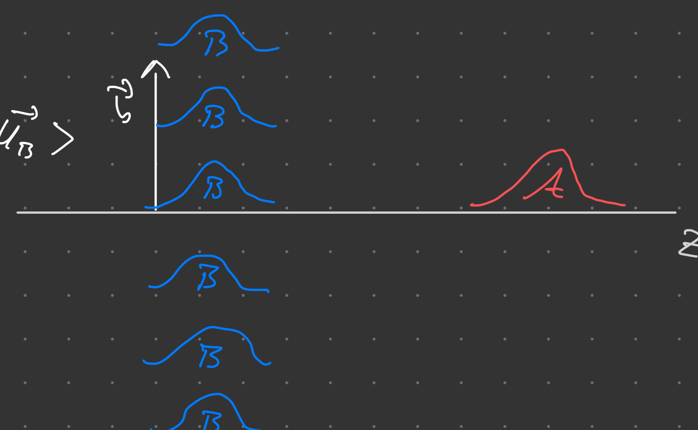
$$P = \left| {}_{out}\langle \phi_1 \dots \phi_n | \phi_A \phi_B \rangle_{in} \right|^2 \;\hat{=}$$



- $|\phi_A \phi_B\rangle_{in}$: in-state at $T \to -\infty$ of two separated wavepackets
- $|\phi_1 \dots \phi_n\rangle_{out}$: out-state at $T \to +\infty$ of n separated wavepackets

3. Fourier transform in-states:

$$|\phi_A \phi_B(\vec{b})\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{\phi_A(\vec{k}_A)\, \phi_B(\vec{k}_B)\, e^{-i\vec{b}\vec{k}_B}}{\sqrt{2E_{k_A} 2E_{k_B}}} \, |\vec{k}_A \vec{k}_B\rangle \qquad (4.9)$$



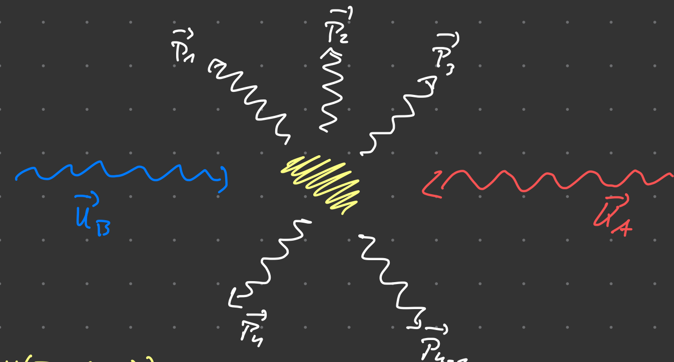
$\vec{b}$: impact parameter

4. Simplification:

$$|\phi_1 \dots \phi_n\rangle_{out} \longrightarrow |\vec{p}_1, \dots, \vec{p}_n\rangle_{out}$$

With (4.9), we are interested in

$${}_{out}\langle \vec{p}_1 \dots \vec{p}_n | \vec{k}_A \vec{k}_B \rangle_{in} \quad \widehat{=}$$



5. S-matrix

$$e^{-iH(T-(-T))}$$

$$\boxed{{}_{out}\langle \vec{p}_1 \dots \vec{p}_n | \vec{k}_A \vec{k}_B \rangle_{in} := \lim_{T\to\infty} \langle \vec{p}_1 \dots \vec{p}_n | e^{-iH2T} | \vec{k}_A \vec{k}_B \rangle \equiv \langle \vec{p}_1 \dots \vec{p}_n | S | \vec{k}_A \vec{k}_B \rangle}$$

Example: $S = \mathbb{1}$ for free theory

6. T-matrix

$$\boxed{S \equiv \underbrace{\mathbb{1}}_{\text{particles miss each other}} + \underbrace{iT}_{\text{non-trivial scattering}}}$$

7. 4-momentum conservation $\rightarrow$

$$\boxed{\langle \vec{p}_1 \ldots \vec{p}_n | iT | \vec{k}_A \vec{k}_B \rangle = \underbrace{(2\pi)^4 \, \delta^{(4)}\Big(k_A + k_B - \sum_f p_f\Big)}_{\text{kinematics}} \times \underbrace{i \underbrace{\mathcal{M}(k_A k_B \rightarrow \{p_f\})}_{\text{"invariant matrix element"}}}_{\text{dynamics}}}$$

- All 4-momenta are on-shell, $p^0 = E_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}$
- Invariant matrix element $\hat{=}$ scattering amplitude of single-particle quantum mechanics

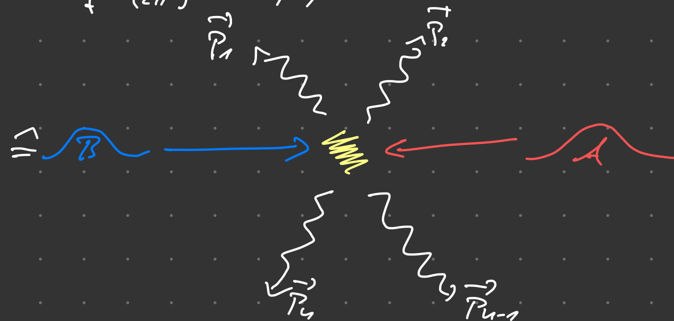
Two questions:

a) $\mathcal{M} = ?$ ($\Rightarrow$ later)

b) $\sigma \overset{?}{=} \sigma(\mathcal{M})$ ($\Rightarrow$ now)

8. $\measuredangle$ Probability to scatter in infinitesimal momentum volume $dV_p = \prod_f d^3 p_f$:

$$dP(AB_{\vec{b}} \mapsto 1 \ldots n) = \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \right) \left| {}_{out}\langle \vec{p}_1 \ldots \vec{p}_n | \phi_A \phi_B(\vec{b}) \rangle_{in} \right|^2$$



9. $\measuredangle$ Single target particle A and many incident particles $B_{\vec{b}}$:

$$d(\#\text{scattering events}) = \int d^2 b \; n_B \; dP(AB_{\vec{b}} \mapsto 1 \ldots n)$$

n_B : B-particles per unit area

By Assumption : $n_B \approx$ const. on interaction length scale l_0

$\longrightarrow \; dP \approx 0 \quad |\vec{b}| \gg l_0$

$\rightarrow$

$$d\sigma = \frac{d(\#\text{ scattering events})}{\rho_B l_B \rho_A l_A \cdot A} = \frac{d(\#\text{ scattering events})}{n_B \cdot 1} = \int d^2b\, dP(AB_{\vec{b}} \mapsto 1\ldots n)$$

$\rho_B \cdot l_B$ $\quad N_A = \rho_A l_A A$

insert everything

$$= \left(\prod \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_{\vec{p}_f}}\right) \int d^2b \prod_{i=A,B} \left(\int \frac{d^3k_i}{(2\pi)^3} \frac{\phi_i(\vec{k}_i)}{\sqrt{2E_{\vec{k}_i}}} \int \frac{d^3q_i}{(2\pi)^3} \frac{\phi^*(\vec{q}_i)}{\sqrt{2E_{\vec{q}_i}}} \right) \underbrace{e^{i\vec{b}(\vec{q}_B - \vec{k}_B)}}_{(2\pi)^2 \delta^{(2)}(k_B^\perp - q_B^\perp)}$$

$k_B^\perp = (k_B^x, k_B^y)$

$$\times \underbrace{\left({}_{out}\langle\{p_f\} | \{\vec{k}_i\}\rangle_{in}\right)}_{i\mathcal{M}(\{k_i\} \to \{p_f\}) \times (2\pi)^4 \delta^{(4)}(\sum_i k_i - \sum_f p_f)} \times \underbrace{\left({}_{out}\langle\{p_f\} | \{\vec{q}_i\}\rangle_{in}\right)^*}_{-i\mathcal{M}^*(\{q_i\} \to \{p_f\}) \times (2\pi)^4 \delta^{(4)}(\sum_i q_i - \sum_f p_f)}$$

$\rightarrow$ Evaluate the 6 q_i-integrals:

a) $q_B^\perp = (q_B^x, q_B^y)$ - integrals $\Rightarrow$ $q_B^\perp = k_B^\perp$

b) $q_A^\perp = (q_A^x, q_A^y)$ - integrals $\Rightarrow$ $q_A^\perp = k_A^\perp$

c) $\times$ $q_A^z\, q_B^z$ - integrals:

depend on $q^z_{A,B}$

$$\int dq_A^z\, dq_B^z \;\delta\left(q_A^z + q_B^z - \sum_f p_f^z\right) \delta\left(E_A + E_B - \sum_f E_f\right) \dots$$

$$= \int dq_A^z \;\delta\Big(\underbrace{\sqrt{\vec{q}_A^{\,2} + m_A^2} + \sqrt{\vec{q}_B^{\,2} + m_B^2} - \sum E_f}_{g(q_A^z)}\Big)\Bigg|_{q_B^z = \sum p_f^z - q_A^z}$$

$$= \frac{1}{|g'(q_A^z)|}\Bigg|_{g(q_A^z)=0} \;\hat{=}\; \frac{1}{\left|\dfrac{q_A^z}{E_A} - \dfrac{q_B^z}{E_B}\right|} = \frac{1}{|V_A - V_B|}$$

where a), b), and $q_B^z = \sum_f p_f^z - q_A^z$ and q_A^z is solution of $g(q_A^z) = 0$

$\Leftrightarrow \; E_A + E_B = \sum_f E_f$

$$\vec{V}_{group} = \frac{\partial E(\vec{q})}{\partial \vec{q}} = \frac{\vec{q}}{E(\vec{q})}$$

10. $\nexists$ $\phi_i(\vec{k}_i)$ peaked around $\vec{p}_i$ for $i = A, B$

$$\rightarrow \quad d\sigma = \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_{p_f}} \right) \frac{|\mathcal{M}(p_A p_B \mapsto \{p_f\})|^2}{2E_{\vec{p}_A} 2E_{\vec{p}_B} |v_A - v_B|}$$

$$\times \int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 k_B}{(2\pi)^3} |\phi_A(\vec{k}_A)|^2 |\phi_B(\vec{k}_B)|^2 (2\pi)^4 \delta^{(4)}\left(k_A + k_B - \sum_f p_f\right)$$

11. $k_A + k_B \approx p_A + p_B$ (Particle detectors cannot resolve momentum spread of initial wave packets)

$$\boxed{d\sigma = \frac{1}{2E_{\vec{p}_A} 2E_{\vec{p}_B} |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_{\vec{p}_f}} \right) |\mathcal{M}(p_A p_B \mapsto \{p_f\})|^2 (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum_f p_f\right)}$$

12. $\oint$ Two final particles (p_1 and p_2) in center-of-mass frame $(\vec{p}_A + \vec{p}_B = 0 \Leftrightarrow \vec{p}_1 = -\vec{p}_2)$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{cm} \hat{=} \frac{1}{2E_{\vec{p}_A} 2E_{\vec{p}_B} |v_A - v_B|} \frac{|\vec{p}_1|}{(2\pi)^2 4 E_{cm}} |\mathcal{M}(p_A p_B \mapsto p_1 p_2)|^2}$$

$E_{cm} = [E_{\vec{p}_A} + E_{\vec{p}_B}]_{cm} = \sqrt{(p_A + p_B)^2}$: center-of-mass energy (Lorentz invariant!)

13. If, in addition, $m_A = m_B = m_1 = m_2$:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{cm} \hat{=} \frac{|\mathcal{M}(p_A p_B \longmapsto p_1 p_2)|^2}{64\pi^2 E_{cm}^2}}$$