4.6. Computing S-Matrix Elements from Feynman Diagrams

Motivation

1. We want $\langle \vec{p}_1 \dots \vec{p}_n | S | \vec{p}_A \vec{p}_B \rangle = \lim_{T\to\infty} \langle \vec{p}_1 \dots \vec{p}_n | e^{-iH(2T)} | \vec{p}_A \vec{p}_B \rangle$

2. Problem:

$$|\vec{p}_A \vec{p}_B\rangle_0 = \sqrt{2E_{\vec{p}_A}}\sqrt{2E_{\vec{p}_B}}\, a^\dagger_{\vec{p}_A} a^\dagger_{\vec{p}_B} |0\rangle \qquad \text{Eigenstate of } H_0$$

$$|\vec{p}_A \vec{p}_B\rangle = ??? \qquad |\vec{p}\rangle \text{ Eigenstate of } H = H_0 + H_{int}$$

3. Remember: For the vacuum found

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

4. Assume it holds similarly

$$|p_A p_B\rangle = \lim_{T\to\infty(1-i\varepsilon)} \underbrace{(\,?\,?\,)}_{\text{Prefactors}\dots} e^{-iHT}|p_A p_B\rangle_0$$

5. If this holds, we could write

$$\langle \vec{p}_1 \ldots \vec{p}_n | S | \vec{p}_A \vec{p}_B \rangle \propto \lim_{T \to \infty(1-i\varepsilon)} {}_0\langle \vec{p}_1 \ldots \vec{p}_n | e^{-iH(2T)} | \vec{p}_A \vec{p}_B \rangle_0$$

Recall: • $U(T, -T) \overset{\text{def}}{=} \mathcal{T} \exp\left[-i \int_{-T}^{T} dt\, H_I(t)\right] \overset{\circ}{=} e^{iH_0(T-t_0)} e^{-iH(2T)} e^{-iH_0(-T-t_0)}$

• $e^{iH_0 T} |\vec{p}_A \vec{p}_B\rangle_0 = e^{i(E_A+E_B)T} |\vec{p}_A \vec{p}_B\rangle_0$

$$\propto \lim_{T \to \infty(1-i\varepsilon)} {}_0\langle p_1 \ldots p_n | \mathcal{T} \exp\left[-i \int_{-T}^{T} dt\, H_I(t)\right] | \vec{p}_A \vec{p}_B \rangle_0$$

6. Correct result:

$$\boxed{\langle \vec{p}_1 \ldots \vec{p}_n | iT | \vec{p}_A \vec{p}_B \rangle = \lim_{T \to \infty(1-i\varepsilon)} \left\{ {}_0\langle \vec{p}_1 \ldots \vec{p}_n | \mathcal{T} \exp\left[-i \int_{-T}^{T} dt\, H_I(t)\right] | \vec{p}_A \vec{p}_B \rangle_0 \right\}_{fc+a}} \quad (4.11)$$

fc+a = "fully connected + amputated"

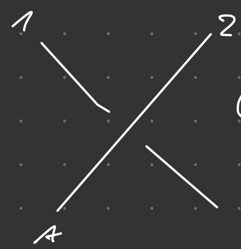
Interpretation & Application

Here: ϕ^4-theory Details: $\Rightarrow$ Problemset 7

1. λ^0-order:

$$ {}_0\langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle_0 = \sqrt{2E_{p_1} 2E_{p_2} 2E_{p_A} 2E_{p_B}} \langle 0 | a_{\vec{p}_1} a_{\vec{p}_2} a^\dagger_{\vec{p}_A} a^\dagger_{\vec{p}_B} | 0 \rangle $$

$$ \stackrel{\circ}{=} 2E_{\vec{p}_A} 2E_{\vec{p}_B} (2\pi)^6 \left\{ \delta^{(3)}(\vec{p}_A - \vec{p}_1)\delta^{(3)}(\vec{p}_B - \vec{p}_2) + \delta^{(3)}(\vec{p}_A - \vec{p}_2)\delta^{(3)}(\vec{p}_B - \vec{p}_1) \right\} $$

$$ = $$  $$ + $$  (4.12)

→ State does not change

→ contributes to $\mathbb{1}$ in $S = \mathbb{1} + iT$

(→ not part of fctn diagrams)

2. λ^1-order:

a) $${}_0\langle \vec p_1 \vec p_2 | \left(-i\frac{\lambda}{4!} \int d^4x \, T\{\phi_I^4(x)\} \right) | \vec p_A \vec p_B \rangle_0$$

Wick's theorem

$$= {}_0\langle \vec p_1 \vec p_2 | \left(-i\frac{\lambda}{4!} \int d^4x \, : \phi_I^4(x) + \text{contraction} : \right) | \vec p_A \vec p_B \rangle_0 \qquad (4.13)$$

b) <u>Careful:</u> Not only full contractions survive because

$$\phi_I^+(x) | \vec p \rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec k}}} a_{\vec k} e^{-ikx} \sqrt{2E_{\vec p}} \, a_p^\dagger |0\rangle = e^{-ipx} |0\rangle$$

$${}_0\langle \vec p | \phi_I^-(x) = \; - - - - \; \overset{\circ}{=} \langle 0 | e^{+ipx}$$

c) <u>Definition:</u> $(| \vec p \rangle_0 \to | \vec p \rangle)$

$$\overline{\phi_I(x) | \vec p} \rangle \equiv e^{-ipx} |0\rangle \quad \hat{=}$$

$$\langle \overline{\vec p | \phi_I}(x) \equiv \langle 0 | e^{+ipx} \quad \hat{=}$$

$$\langle \overline{\vec p | \vec q} \rangle \equiv 2 E_{\vec p} (2\pi)^3 \delta^{(3)}(\vec p - \vec q) \quad \hat{=}$$

d) Then

$$\boxed{{}_0\langle \vec{p}_1 \dots | T\{\phi_a \dots\} | \vec{p}_A \dots \rangle_0 = \left\{ \begin{array}{l} \text{Sum of all full contractions of} \\ \text{fields and external state momenta} \end{array} \right\}}$$

Example:

$${}_0\langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle_0 = \langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle + \langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle$$

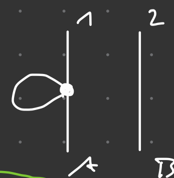
$$= (4.12)$$

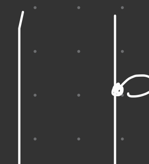
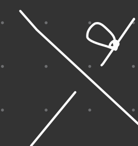
e) Application to (4.13): $\quad -i\frac{\lambda}{4!}\int d^4x\ {}_0\langle \vec{p}_1 \vec{p}_2 | T\{\phi_I(x)\} | \vec{p}_A \vec{p}_B\rangle_0$

$=$ 

$+$ 

$\langle p_1 p_2 | \phi\phi\phi\phi | p_A p_B \rangle \qquad \langle p_1 p_2 | \phi\phi\phi\phi | p_A p_B \rangle$

$+$ 

$+$ 
$+$ 


$\langle p_1 p_2 | \phi\phi\phi\phi | p_A p_B \rangle$

... ...

$+$ 

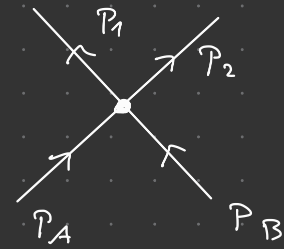
$\langle p_1 p_2 | \phi\phi\phi\phi | p_A p_B \rangle$

→ Terms with $\overline{\phi\phi}\,\overline{\phi\phi}$ and $\overline{\phi\phi}\phi\phi$ do not contribute to T

→ Only fully connected diagrams contribute to T

f) Therefore

$$\langle \vec{p}_1 \vec{p}_2 | iT | \vec{p}_A \vec{p}_B \rangle \approx$$



$$= 4! \left(-i\frac{\lambda}{4!}\right) \int d^4x \, e^{-i(p_A + p_B - p_1 - p_2)x}$$

$$= -i\lambda \, (2\pi)^4 \, \delta^{(4)}(p_A + p_B - p_1 - p_2)$$

$$\overset{\text{def.}}{=} i\mathcal{M}(p_A p_B \mapsto p_1 p_2) \, (2\pi)^4 \, \delta^{(4)}(p_A + p_B - p_1 - p_2)$$

$$\rightarrow \quad \mathcal{M}(p_A p_B \mapsto p_1 p_2) = -\lambda + \mathcal{O}(\lambda^2)$$

$\rightarrow$ (Problemset 7)

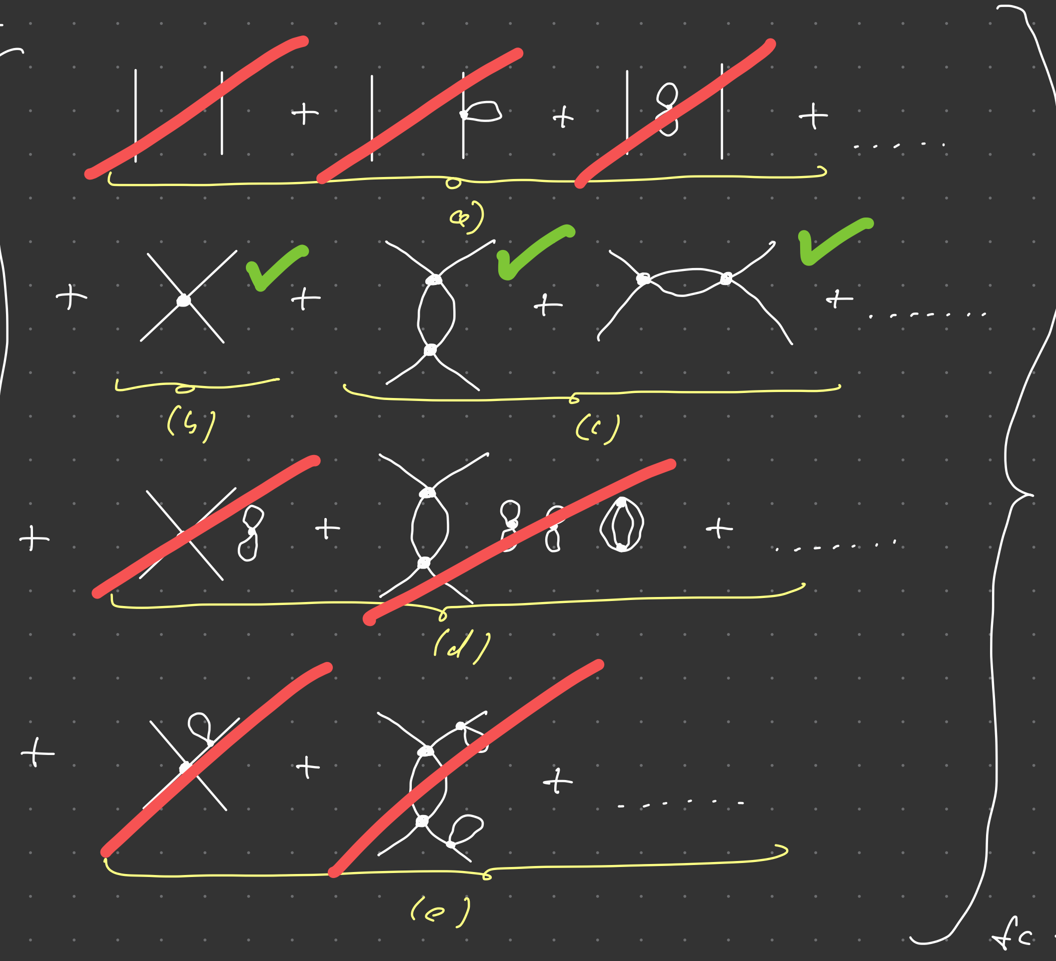
$$\sigma_{total} = \frac{\lambda^2}{32\pi E_{cm}^2}$$

factor of $\frac{1}{2}$

$$|p_1 p_2\rangle = |p_2 p_1\rangle$$

3. Higher-order contributions:

$\langle \vec{p}_1 \vec{p}_2 | iT | \vec{p}_A \vec{p}_B \rangle =$



Which of these diagrams are "fc+a"?

a) Not fully connected ✗

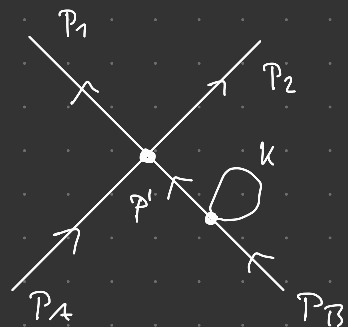
b) λ^1-order contribution ✓

c) Higher-order contributions ✓

d) Diagrams with vacuum bubbles
→ bubbles exponentiate and drop out (as before) ✗

e) Fully connected diagrams with "appendices" to external legs ✗ ✓ ?

fc + a



$$\hat{=} \frac{1}{2} \int \frac{d^4p'}{(2\pi)^4} \frac{i}{(p'^2 - m^2)} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2}$$

$$\times (-i\lambda)(2\pi)^4 \delta^{(4)}(p_A + p' - p_1 - p_2)$$

$$\times (-i\lambda)(2\pi)^4 \delta^{(4)}(p_B - p')$$

$$\sim \frac{1}{p_B^2 - m^2} = \frac{1}{0} = \infty$$

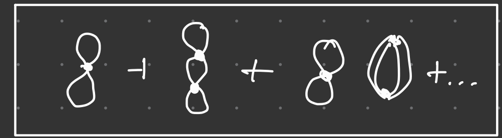
(on-shell: $p_B^2 = m^2$

$\rightarrow$ (4.11) makes only sense without these diagrams!

Interpretation:

Vacuum $|0\rangle$

+

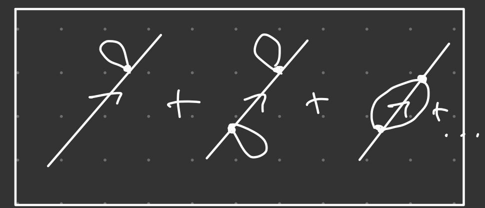


Interacting vacuum $|\Omega\rangle$

$$\xrightarrow{+H_{int}}$$

Single-particle state $|\vec{p}\rangle_0$

+



interacting single-particle state $|\vec{p}\rangle$

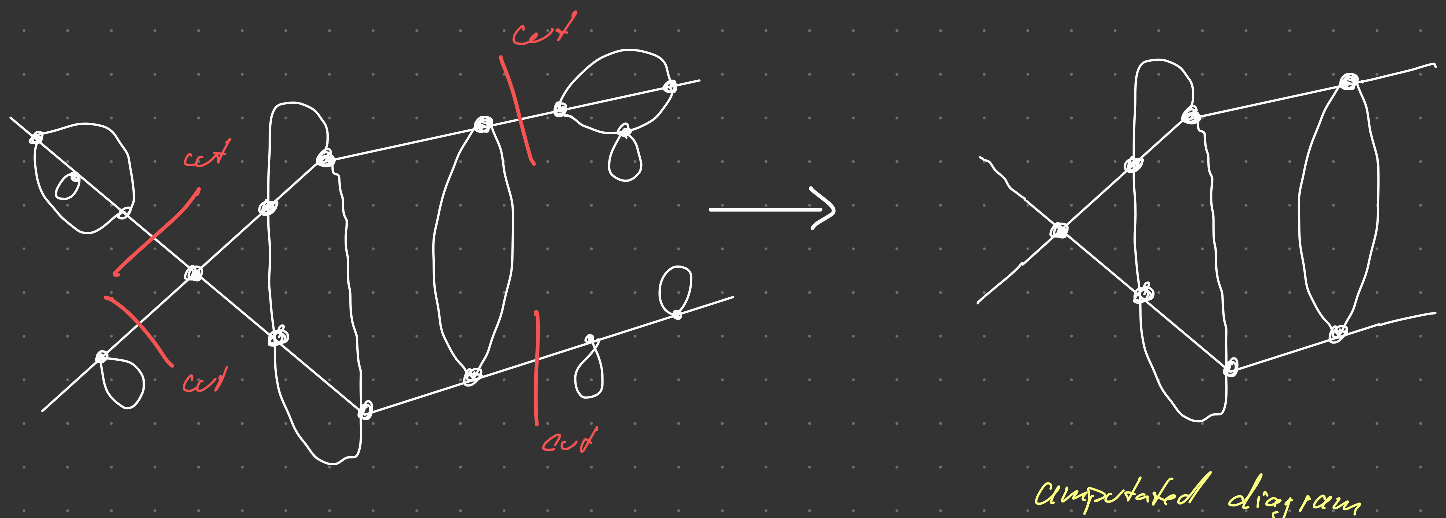
$\longrightarrow$ not related to scattering!

$\longrightarrow$

4. Amputation of diagrams:

Starting from tip of each external leg, cut at the last point at which the diagram can be cut by removing a single propagator, such that this operation separates the leg from the rest of the diagram.

Example:

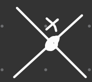


5. $\rightarrow$

$$(4.11) = i\mathcal{M}\,(2\pi)^4\,\delta^{(4)}\Big(p_A + p_B - \sum_f p_f\Big)$$

$$= \left\{ \begin{array}{l} \text{Sum of all fully connected, amputated} \\ \text{Feynman diagrams with } p_A, p_B \text{ incoming} \\ \text{and } \{p_f\} \text{ outgoing} \end{array} \right\}$$

6. $\rightarrow$ Position-space Feynman rules for scattering amplitudes in ϕ^4-theory:

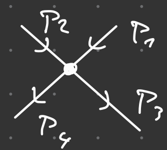
1. For each edge, $x \text{———} y \;=\; D_F(x-y)$
2. For each vertex,  $= (-i\lambda)\int d^4x$
3. For each external line, $x \text{—}\!\leftarrow\!\text{—} p \;=\; e^{-ipx}$
4. Divide by symmetry factor, $\frac{1}{S} \times \ldots$

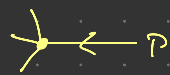
7. ∡ Momentum-space representation of D_F + vertex integration

→ δ-distributions at vertices + momentum integrals

→ <u>Momentum-space Feynman rules</u> for scattering amplitudes in ϕ^4-theory:

1. For each edge,  $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

2. For each vertex,  $= (-i\lambda)\,(2\pi)^4\,\delta(p_1 + p_2 - p_3 - p_4)$

3. For each external line,  $= 1$

5. Integrate over internal momenta, $\prod_i \int \dfrac{d^4 p_i}{(2\pi)^4}$

6. Divide by sym. factor, $\dfrac{1}{S} \times \ldots$

Note: