4.7. Feynman Rules for Quantum Electrodynamics

1. Fields:

$$\boxed{\begin{array}{lll} \text{Fermions:} & \psi(x) & (\text{bispinor field}) \\ \text{Photons:} & A_\mu(x) & (\text{vector field}) \end{array}}$$

2. Lagrangian

$$\begin{aligned} \mathcal{L}_{QED} &= \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{int} \\ &= \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \underbrace{e\,\bar{\psi}\gamma^\mu\psi}_{j^\mu} A_\mu \\ &= \bar{\psi}(iD\!\!\!\!/ - m)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \end{aligned}$$

m: mass of fermions

e: charge of the fermions (= coupling constant)

D_μ: covariant derivative: $D_\mu = \partial_\mu + ieA_\mu$

3. Hamiltonian

$$\boxed{\begin{aligned} &H_{QED} = H_{Dirac} + H_{Maxwell} + H_{int} \\ &\text{with} \quad H_{int} = e \int d^3x \, \bar{\psi} \gamma^\mu \psi A_\mu \end{aligned}}$$

4. Equation of motion:

$$\boxed{\begin{aligned} &(i\not{D} - m)\psi = 0 \qquad \text{(gauge-covariant Dirac equation)} \\ &\partial_\nu F^{\nu\mu} = j^\mu \qquad \text{(inhomogeneous Maxwell equation)} \end{aligned}}$$

Note 4.2 $\mathcal{L}_{QED}$ is invariant under $U(1)$ gauge transformations,

$$\psi'(x) = e^{ie\alpha(x)} \psi(x)$$

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x)$$

for arbitrary $\alpha : \mathbb{R}^{1,3} \to \mathbb{R}$

Note 4.3

The QED-sector of the standard model (SM) includes several copies of the fermion field that all couple to the same photon field,

$$\mathcal{L}^{SM}_{QED} = \sum_f \left\{ \bar{\psi}_f (i\not{\partial} - m_f)\psi_f - q_f \bar{\psi}_f \gamma^\mu \psi_f A_\mu \right\} - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

with mass m_f and charge q_f of fermion type

$$f \in \{\text{Leptons}, \text{Quarks}\} = \{e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, c, s, t, b\}$$

Notes on the Fermion / Dirac Sector

Remember:

$$S_F^{ab}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\varepsilon} e^{-ip(x-y)}$$

$$= \begin{cases} \langle 0 | \psi_a(x) \overline{\psi}_b(y) | 0 \rangle & \text{for } x^0 > y^0 \\ -\langle 0 | \overline{\psi}_b(y) \psi_a(x) | 0 \rangle & \text{for } x^0 < y^0 \end{cases}$$

$$\equiv \langle 0 | T \psi_a(x) \overline{\psi}_b(y) | 0 \rangle \qquad (4.16)$$

To deal with H_{int} perturbatively, we need Wick's theorem for fermions:

1. Time ordering: (4.16) suggest for $\psi \in \{\psi, \overline{\psi}\}$

$$\boxed{T\{\psi_{\sigma 1} \ldots \psi_{\sigma n}\} \equiv (-1)^{\#} \psi_1 \ldots \psi_n \quad \text{for } x_1^0 > x_2^0 > \ldots > x_n^0}$$

σ = permutation of $\{1, 2, \ldots n\}$

$(-1)^{\#}$: signum of σ with # number of operator interchanges

2. Normal order:

Define for $x \in \{a^s_{\vec{p}}, b^s_{\vec{p}}, a^{s\dagger}_{\vec{p}}, b^{r\dagger}_{\vec{p}}\}$

$$\boxed{:x_1 \dots x_n: \equiv (-1)^{\#} (\text{creation operators}) \times (\text{annihilation operators})}$$

$\#$: number of operator interchanges

3. Contraction:

Define

$$\boxed{\overbrace{\psi_a(x)\psi_b(y)} \equiv T\{\psi_a(x)\psi_b(y)\} - :\psi_a(x)\psi_b(y):}$$

$$\boxed{\begin{aligned}
\overbrace{\psi_a(x)\bar{\psi}_b(y)} &\stackrel{\circ}{=} \begin{cases} \{\psi_a^+(x), \bar{\psi}_b^-(y)\} & \text{for } x^0 > y^0 \\ -\{\bar{\psi}_b^+(y), \psi_a^-(x)\} & \text{for } x^0 < y^0 \end{cases} \stackrel{\circ}{=} S_F^{ab}(x-y) \\
\overbrace{\psi_a(x)\psi_b(y)} &\stackrel{\circ}{=} 0 \qquad \leftarrow \text{ use } \{a^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}\} = 0 \\
\overbrace{\bar{\psi}_a(x)\psi_b(y)} &\stackrel{\circ}{=} 0
\end{aligned}}$$

4. Contraction & Normal order:

$$: A\, \psi_a(x)\, B\, \psi_b(y)\, C : \;\equiv\; (-1)^{\#}\, \psi_a(x)\, \psi_b(y) \cdot :ABC:$$

$\#$: number of operator interchanges

5. Wick's theorem: $\psi \in \{\Psi, \bar{\Psi}\}$

$$T\{\psi_a(x_1)\, \psi_b(x_2) \ldots\} = :\psi_a(x_1)\, \psi_b(x_2) \ldots + \text{all possible contractions}:$$

Notes on the Photon / Maxwell Sector

1. Observation: A^μ has four degrees of freedom but there are only two photon polarizations!

2. Problem: Gauge invariance

→ Unphysical degrees of freedom

→ Fix gauge to quantize only physical degrees of freedom

3. Different solutions:

- Coulomb gauge $\nabla \cdot \vec{A} = 0$ (⇒ Advanced quantum mechanics)
- Lorenz gauge $\partial_\mu A^\mu = 0$ (Gupta-Bleuler formalism, ⇒ Itzykson & Zuber, pp. 127-134)
- Faddeev-Popov procedure (⇒ later)

Note: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \rightarrow \Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0} \Rightarrow \Pi^0 = F^{00} = 0$

$$\rightarrow [A_\mu(x), \Pi_0(y)] \overset{!}{=} i g_{\mu\nu} \delta^{(3)}(\vec{x} - \vec{y})$$

4. Motivation

a) $\mathcal{L}$ Lorenz gauge: $\partial_\mu A^\mu = 0 \longrightarrow$ EOM for $\mathcal{L}_{\text{Maxwell}}$: $\partial^2 A^\nu = 0$

b) Expand field in classical solutions:

$$\boxed{A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=0}^{3} \left[a_{\vec{p}}^r \, \varepsilon_\mu^r(p)\, e^{-ipx} + a_{\vec{p}}^{r\dagger} \, \varepsilon_\mu^{r*}(p)\, e^{ipx} \right]}$$

with $p^2 = 0 \iff E_{\vec{p}} = p^0 = |\vec{p}|$

ε_μ^r: Polarization 4-vector

5. Results:

a) Impose constraints on external (physical) photons

$$\boxed{\varepsilon^\mu(p) = \begin{pmatrix} 0 \\ \vec{\varepsilon}(p) \end{pmatrix} \quad \text{and} \quad \vec{p}\cdot\vec{\varepsilon}(p) = 0 \qquad \text{(transverse polarization)}}$$

$\rightarrow$ Two $r, s = 1, 2$ independent bosonic modes for each momentum $\vec{p}$:

$$\boxed{[a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}] = (2\pi)^3 \delta_{rs}\, \delta^{(3)}(\vec{p}-\vec{q}) \quad \text{and} \quad [a^r_{\vec{p}}, a^s_{\vec{q}}] = 0 = [a^{r\dagger}_{\vec{p}}, a^{s\dagger}_{\vec{q}}]}$$

b) Propagator (Feynman gauge):

$$\boxed{\langle 0 | T\{A_\mu(x) A_\nu(y)\} | 0\rangle \overset{*}{=} \int \frac{d^4 q}{(2\pi)^4} \frac{-i g_{\mu\nu}}{q^2 + i\varepsilon} e^{-iq(x-y)}}$$

Feynman rules

1. Expectations:

a) Two fields (ψ_a and A_μ) $\longrightarrow$ Two propagators $\longrightarrow$ Two line-types:

Fermions (with spinor indices a and b): a  b

Photons (with 4-vector indices μ and ν): μ  ν

$\longrightarrow$ Two particle types: (anti) fermions & photons

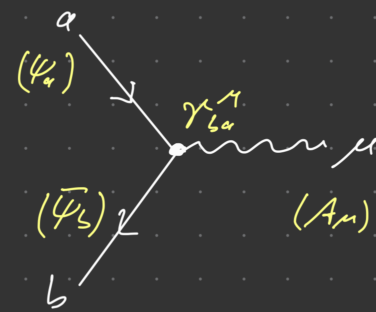
$\longrightarrow$ Two types of external states:

Fermion / Antifermion: $|\vec{p}, s\rangle_{a,b}$ (s: Spin; a: Fermion; b: Antifermion)

Photon: $|\vec{p}, r\rangle$ (r: Polarization)

b) Interaction with <u>three</u> fields ($\mathcal{H}_{int} \sim \bar{\psi}_b \gamma^\mu_{ba} \psi_a \cdot A_\mu$)

→ Vertices of degree 3:



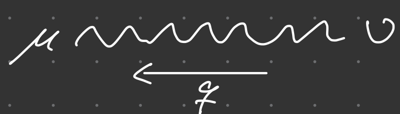
2. Momentum-space Feynman rules (for scattering amplitudes):

(Note: In many textbooks, the colored indices are omitted.)

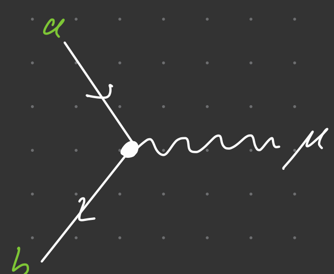
Propagators

Fermions:  $= \dfrac{i(\not{p}+m)_{ba}}{p^2-m^2+i\varepsilon} \;\hat{=}\; \psi_b(x)\,\bar{\psi}_a(y)$

Note:  simplified

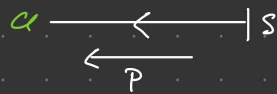
Photons:  $= \dfrac{-i\,g_{\mu\nu}}{q^2+i\varepsilon} \;\hat{=}\; A_\mu(x)\,A_\nu(y)$

Vertices

 $= -ie\,\gamma^{\mu}_{ba} \;\hat{=}\; (-ie)\int d^4z\;\gamma^{\mu}_{ba}$

External legs

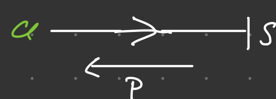
Fermions:



$$= u^s_\alpha(p) \quad \hat{=} \quad \psi_\alpha |\vec{p}, s\rangle_a$$

$$= \bar{u}^s_\alpha(p) \quad \hat{=} \quad \langle \vec{p}, s|_a \, \bar{\psi}_\alpha$$

Antifermion:

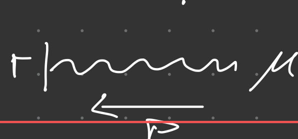


$$= \bar{v}^s_\alpha(p) \quad \hat{=} \quad \bar{\psi}_\alpha |\vec{p}, s\rangle_b$$

$$= v^s_\alpha(p) \quad \hat{=} \quad \langle \vec{p}, s|_b \, \psi_\alpha$$

Photons:

$$= \varepsilon^r_\mu(p) \quad \hat{=} \quad A_\mu |\vec{p}, r\rangle$$



$$= \varepsilon^{r*}_\mu(p) \quad \hat{=} \quad \langle \vec{p}, r| A_\mu$$

Evaluation :

1. Impose momentum conservation at each vertex
2. Integrate over all undetermined momenta
3. Compute the overall sign of the diagram