First application: The Coulomb Potential

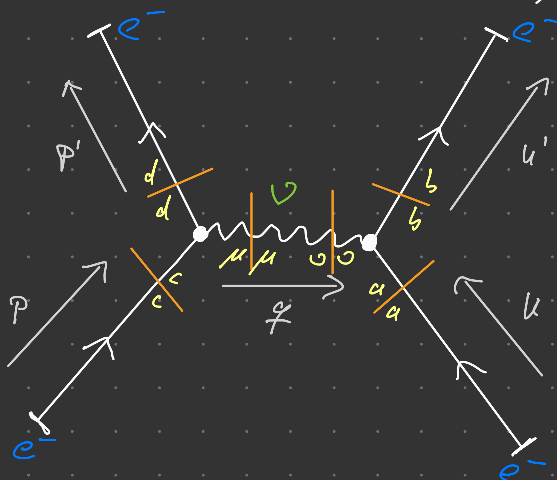
1. Scattering process (Møller scattering)

$$\boxed{\text{Electron } (e^-) + \text{Electron } (e^-) \longrightarrow \text{Electron } (e^-) + \text{Electron } (e^-)}$$

a) Contribution to the tree-level amplitude (sufficient for distinguishable fermions):

$$i\mathcal{M}\left(e^-(p)\,e^-(k) \to e^-(p')\,e^-(k')\right)$$

=



$$= \sigma \cdot \bar{u}_d(p')\left(-ie\gamma^{\mu}_{dc}\right)u_c(p) \times \left(\frac{-ig_{\mu\nu}}{q^2}\right) \times \bar{u}(k'_b)\left(-ie\gamma^{\nu}_{ba}\right)u_a(k)$$

$$= \sigma \cdot \bar{u}(p')(-ie\gamma^{\mu})u(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k')(-ie\gamma^{\nu})u(k)$$

sign of the diagram

$$p - p' = q = k' - k$$

b) Nonrelativistic limit: $|\vec{p}|^2 \ll m^2$

$$u(p) = \begin{pmatrix} \sqrt{p\sigma}\,\xi \\ \sqrt{p\bar{\sigma}}\,\xi \end{pmatrix} \approx \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \quad \text{and} \quad \frac{1}{(p-p')^2} \approx \frac{-1}{|\vec{p}-\vec{p}'|^2}$$

$$" \begin{pmatrix} 0 \\ \vec{p}-\vec{p}' \end{pmatrix}^2 " = -|\vec{p}-\vec{p}'|^2$$

Therefore:

$$\bar{u}(p')\gamma^\mu u(p) \approx \begin{cases} 2m\,\xi^\dagger_{p'}\xi_p & \mu = 0 \\ 0 & \mu = 1,2,3 \end{cases}$$

and

$$i\mathcal{M} \approx \sigma \cdot \frac{-ie^2}{|\vec{p}-\vec{p}'|^2}\left(2m\,\xi^\dagger_{p'}\xi_p\right)\left(2m\,\xi^\dagger_{k'}\xi_k\right) \qquad (4.17)$$

c) Compare with nonrelativistic scattering theory ($\hat{=}$ Born approximation):

$$\langle p'|iT|p\rangle \overset{*}{=} -i\,\hat{V}(\vec{q})\,\delta(E_{\vec{p}'}-E_{\vec{p}}) \qquad (\vec{q} = \vec{p}'-\vec{p})$$

$\hat{V}(\vec{q})$: Fourier transform of static scattering potential

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137} \quad \text{Fine-Structure constant}$$

$$\longrightarrow \quad V(\vec{q}) = \sigma \cdot \frac{e^2}{|\vec{q}|^2} \quad \Longrightarrow \quad V(\vec{r}) \overset{\circ}{=} \sigma \cdot \frac{e^2}{4\pi|\vec{r}|} = \sigma \cdot \frac{\alpha}{r}$$

d) Sign of the diagram:

$$ {}_{ua}\langle \vec{p}'\vec{k}' | \bar{\Psi}\Psi A \bar{\Psi}\Psi A | \vec{p}\vec{k}\rangle_{ua} = \langle 0| a_{\vec{k}'} a_{\vec{p}'} \bar{\Psi}\Psi A \bar{\Psi}\Psi A \, a^{+}_{\vec{p}} a^{+}_{\vec{k}} |0\rangle $$

$\longrightarrow 1 + 1 + 2 = 4$ interchanges $\longrightarrow \sigma = +1$

e) $\longrightarrow$ Repulsive Coulomb potentials:

$$ V_{e^-e^-}(r) = + \frac{e^2}{4\pi r} $$

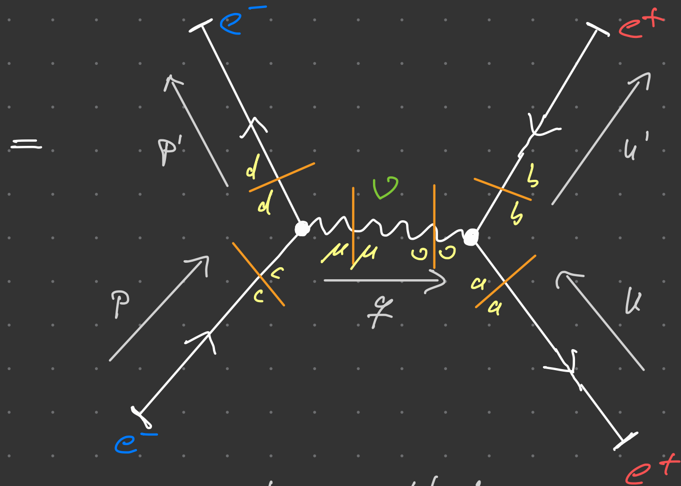
$\longrightarrow$ Equal charges repel each other!

2. γ Scattering process (Bhabha scattering)

$$\boxed{\text{Electron } (e^-) + \text{Positron } (e^+) \longrightarrow \text{Electron } (e^-) + \text{Positron } (e^+)}$$

a) Contribution to the tree-level amplitude:

$$i\mathcal{M}(e^-(p)\,e^+(k) \to e^-(p')\,e^+(k'))$$

$$=$$ 

$$= \sigma \cdot \bar{u}_d(p')(-ie\gamma^\mu_{dc})u_c(p) \times \left(\frac{-ig_{\mu\nu}}{q^2}\right)$$
$$\times\ \bar{v}(k_b)(-ie\gamma^\nu_{ba})v_a(k')$$

$$= \sigma \cdot \bar{u}(p')(-ie\gamma^\mu)u(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{v}(k)(-ie\gamma^\nu)v(k')$$

↑ sign of the diagram

$$p - p' = q = k' - k$$

b) Nonrelativistic limit $\to$ Same as (4.17) (with $k \leftrightarrow k'$), but what is σ?

c) Sign of the diagram:

$$\langle 0|\, b_{\vec{k}'}\, a_{\vec{p}'}\, \bar{\Psi}\Psi A\, \bar{\Psi}\Psi A\, a^\dagger_p\, b^\dagger_k\,|0\rangle \longrightarrow 2+1+2 = 5 \text{ interchanges} \longrightarrow \sigma = -1$$

d) → Attractive Coulomb Potential:

$$V_{e^+e^-}(r) = -\frac{e^2}{4\pi r}$$

→ Opposite charges attract each other

5. Elementary Processes of Quantum Electrodynamics

5.1 Cross Section of $e^+e^- \longrightarrow \mu^+\mu^-$ Scattering

1. ~~X~~ Reaction

$$\boxed{\text{Electron } (e^-) + \text{Positron } (e^+) \longrightarrow \text{Muon } (\mu^-) + \text{Antimuon } (\mu^+)}$$

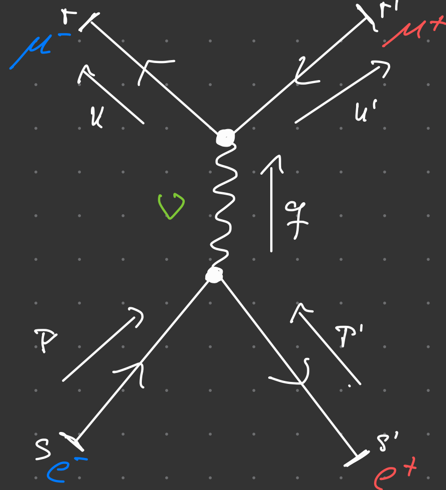
2. Note: Both electrons and muons are spin-$\frac{1}{2}$ fermions with equal charge

$q_e = q_\mu = e = -|e|$ but different mass $m_e \ll m_\mu$:

$$\mathcal{L}^{e,\mu}_{QED} = \sum_{f=e,\mu} \left\{ \bar{\Psi}_f (i\not{\partial} - m_f) \Psi_f - q_f \bar{\Psi}_f \gamma^\mu \Psi_f A_\mu \right\} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

3. Tree-level amplitude:

$$i\mathcal{M}\left(e^-(p)\, e^+(p') \to \mu^-(k)\, \mu^+(k')\right)$$



$$= \bar{v}_e^{s'}(p')(-ig_e\gamma^\mu)u_e^s(p)\left(\frac{-ig_{\mu\sigma}}{q^2}\right)\bar{u}_\mu^r(k)(-ig_\mu\gamma^\sigma)v_\mu^{r'}(k')$$

$$= \frac{ie^2}{q^2}\left(\bar{v}(p')\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v(k')\right)$$

with $p+p'=q=k+k'$

4. We want $d\sigma \propto |\mathcal{M}|^2 \to$ need $\mathcal{M}^*$.

Use $\left(\bar{v}\gamma^\mu u\right)^* \overset{\circ}{=} \left(\bar{u}\gamma^\mu v\right)$

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\underbrace{\left(\bar{v}(p')\gamma^\mu u(p)\,\bar{u}(p)\gamma^\sigma v(p')\right)}_{\square}\left(\bar{u}(k)\gamma_\mu v(k')\,\bar{v}(k')\gamma_\sigma u(k)\right)$$

5. Typical collider setup:

- e^+- and e^--beam unpolarized $\longrightarrow$ average over spin polarization of in-states
- Muon detector cannot resolve spin $\longrightarrow$ sum over spin polarizations of out-states

$\rightarrow$
$$d\sigma \propto \frac{1}{4} \sum_{s,s'} \sum_{r,r'} |\mathcal{M}(s,s' \rightarrow r,r')|^2$$

6. Use spin sums (3.10) and spinor indices to evaluate $\square$:

$$\sum_{ss'} \bar{v}^{s'}_a(p') \gamma^\mu_{ab} u^s_b(p) \bar{u}^s_c(p) \gamma^\nu_{cd} v^{s'}_d(p') \overset{\circ}{=} \mathrm{Tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right]$$

Details: ⊖ Problemset 8

7. $\rightarrow$
$$\frac{1}{4}\sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{4q^4} \mathrm{Tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right] \mathrm{Tr}\left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right]$$

8. Trace technology: (⇒ Problemset 7)

Trace identities:

$$\mathrm{Tr}[\text{odd \# of } \gamma] = 0$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4 g^{\mu\nu}$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4 (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$$

$$\mathrm{Tr}[\gamma^5] = 0$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^5] = 0$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] = -4 i \varepsilon^{\mu\nu\rho\sigma}$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \ldots] = \mathrm{Tr}[\ldots \gamma^\sigma \gamma^\rho \gamma^\nu \gamma^\mu]$$

Contraction identities

$$\gamma^\mu \gamma_\mu = 4$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2 \gamma^\nu$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4 g^{\nu\rho}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2 \gamma^\sigma \gamma^\rho \gamma^\nu$$

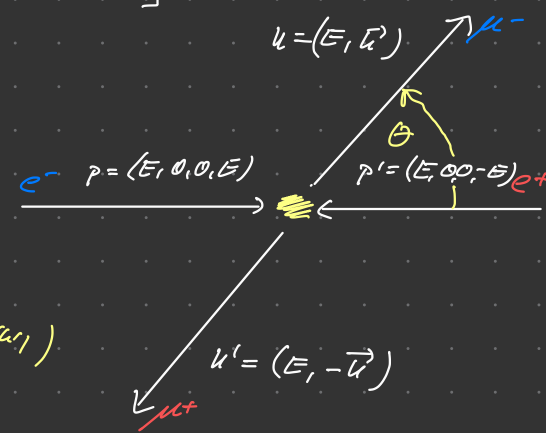
9. $\rightarrow$ $\mathrm{Tr}\left[(\not{p}'-m_e)\gamma^{\mu}(\not{p}+m_e)\gamma^{\nu}\right] \overset{\circ}{=} 4\left[p'^{\mu}p^{\nu}+p'^{\nu}p^{\mu}-g^{\mu\nu}(pp'+m_e^2)\right]$

$\mathrm{Tr}\left[(\not{k}+m_m)\gamma_{\mu}(\not{k}'-m_m)\gamma_{\nu}\right] \overset{\circ}{=} 4\left[k_{\mu}k'_{\nu}+k_{\nu}k'_{\mu}-g_{\mu\nu}(kk'+m_m^2)\right]$

10. Since $m_e/m_m \approx 1/200$, we set $m_e=0$ henceforth:

($\ominus$ Problemset 8 for $m_e \neq 0$)

$$\frac{1}{4}\sum_{s,s',r,r'}|\mathcal{M}|^2 \overset{\circ}{=} \frac{8e^4}{q^4}\left[(pk)(p'k')+(pk')(p'k)+m_m^2(pp')\right]$$



11. $\measuredangle$ Center-of-mass frame: $\vec{p}+\vec{p}\,'=0=\vec{k}+\vec{k}'$

- W.l.o.g. $p=(E,E\hat{z})$, $p'=(E,-E\hat{z})$ $\quad (m_e=0)$
- $|\vec{k}| = \sqrt{E^2-m_m^2}$ $\quad \left(E=E_e(p)=E_e(p')=E_m(k)=E_m(k')\right)$
- $\vec{k}\cdot\hat{z} = |\vec{k}|\cdot\cos\theta$

$\longrightarrow \quad q^2 = (p+p')^2 = 4E^2$

$$pp' = 2E^2$$

$$pk = p'k' = E^2 - E|\vec{k}|\cos\theta$$

$$pk' = p'k = E^2 + E|\vec{k}|\cos\theta$$

$\longrightarrow \quad \overline{|\mathcal{M}|^2} \equiv \frac{1}{4}\sum_{ss'rr'} |\mathcal{M}|^2$

$$\hat{=} \; e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

12. Differential scattering cross section from (4.12):

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{\underset{\substack{\shortparallel\\ E}}{2E_p}\,\underset{\substack{\shortparallel\\ E}}{2E_{p'}}\,\underbrace{|\underset{\substack{\shortparallel\\ \frac{p^3}{E_p}=1}}{v_p} - \underset{\substack{\shortparallel\\ -1}}{v_{p'}}|}_{2}} \; \frac{|\vec{k}|}{(2\pi)^2\, 4\underset{\substack{\shortparallel\\ 2E}}{E_{cm}}} \; \overline{|\mathcal{M}|^2}$$

$$= \frac{\alpha^2}{2E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

13. Total cross section:

$$\boxed{\sigma_{total} = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{m_\mu^2}{2E^2}\right)}$$

14. Discussion

- For $E_{cm} < 2m_\mu$ no pair-production is possible
- Prediction of QED: non-trivial energy dependence of $\overline{\sigma}$

  $\Leftrightarrow$ Experimental results verify this additional dependence!
- Measuring σ_{total} as a function of E_{cm} yields the muon mass m_μ.

5.2. Summary of QED calculations

1. Draw relevant Feynman diagrams.

2. Use Feynman rules to calculate $\mathcal{M}$.

3. Calculate $|\overline{\mathcal{M}}|^2 = \sum_{\text{spins}} |\mathcal{M}|^2$ (use spin-sum relations).

4. Evaluate traces (use trace technology).

5. Fix a frame of reference and express all 4-momenta in terms of kinematic variables (energies, angles....)

6. Plug in $|\overline{\mathcal{M}}|^2$ in Eq. (4.11) and integrate over phase-space variables that are not measured.