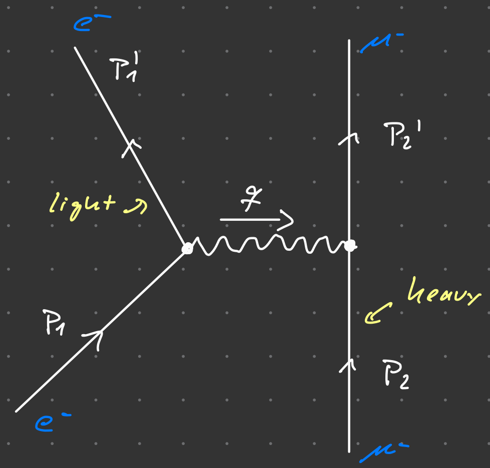
6. Radiative Corrections of QED

6.1. Overview

1. Process: For simplicity, e^- scattering of a very heavy particle, e.g.,

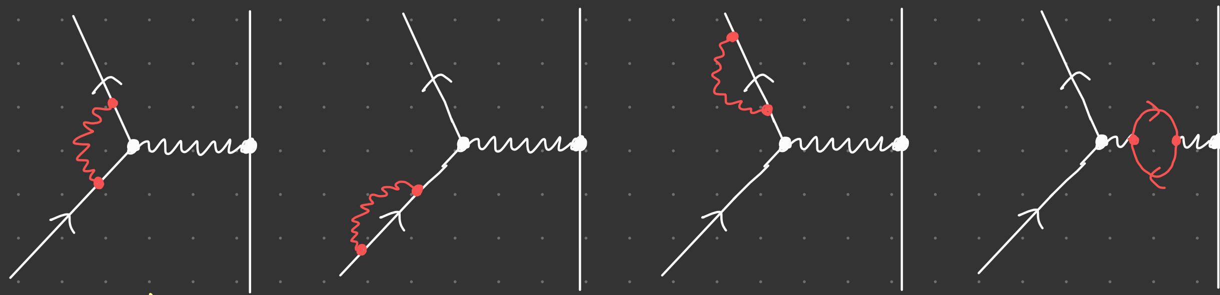
$$\boxed{\lim_{m_\mu \to \infty} \left\{ \text{Electron } (e^-) + \text{Muon } (\mu^-) \longrightarrow \text{Electron}(e^-) + \text{Muon } (\mu^-) \right\}}$$

2. Tree-level:



3. Radiative corrections = Higher-order contributions to tree-level amplitudes from diagrams with...

- loops:



(a) (b) (c)

a) Vertex correction: UV-divergence & IR-divergence

(most interesting, ⇒ below)

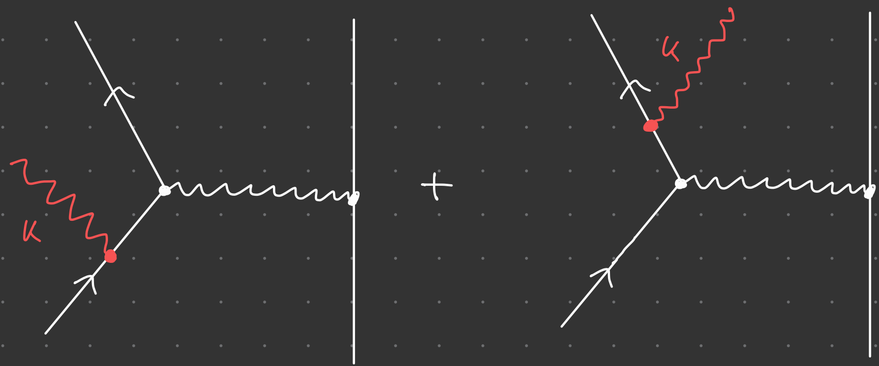
b) External leg corrections: UV-divergence & IR-divergence

(not computated, ⇒ later)

c) Vacuum polarization: UV-divergence

(complicated evaluation, ⇒ later)

- extra final-state photons (Bremsstrahlung)



$\longrightarrow$ IR-divergence $k \to 0$

4. Spoilers:

- UV-divergences: cancel in observable quantities
- IR-divergences: cancel with the divergences of the Bremsstrahlung diagrams

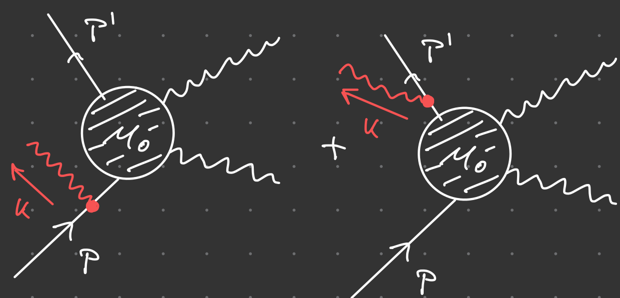
6.2 Soft Bremsstrahlung

1. <u>Bremsstrahlung</u> = Electromagnetic radiation emitted by decelerated, charged particles

   <u>Soft</u> = Low-energy photons ($k \approx 0$)

2. Can be classically derived from Maxwell's equations (→ PES pp. 177–182)

3. $\exists$ Corresponding QFT processes:



$$= i\mathcal{M} = -ie\,\epsilon^*_\mu(k)\,\bar{u}(p')\left\{ \mathcal{M}_0(p', p-\cancel{k})\,\frac{i(\not p - \cancel{\not k} + m)}{(p-k)^2 - m^2 + i\epsilon}\,\gamma^\mu + \gamma^\mu\,\frac{i(\not p' + \cancel{\not k} + m)}{(p'+k)^2 - m^2 + i\epsilon}\,\mathcal{M}_0(p'\cancel{+k}, p) \right\} u(p)$$

$\mathcal{M}_0$: (unknown) interaction amplitude

4. Simplifications

- Soft photons: $|\vec{k}| \ll |\vec{p}' - \vec{p}|$

$\rightarrow \mathcal{M}_0(p', p-k) \approx \mathcal{M}_0(p'+k, p) \approx \mathcal{M}_0(p', p)$

$\rightarrow \not{p} - \not{k} \approx \not{p}$ etc.

- Dirac algebra: $\longrightarrow$

Use $[\,(\not{p} - m)\, u(p) = 0\,]$

$$(\not{p} + m)\gamma^\mu \varepsilon_\mu^* u(p) \overset{\circ}{=} 2p^\mu \varepsilon_\mu^* u(p)$$

$$\bar{u}(p')\gamma^\mu \varepsilon_\mu^* (\not{p}' + m) \overset{\circ}{=} \bar{u}(p')\, 2p'^\mu \varepsilon_\mu^*$$

- $$\left.\begin{aligned} (p-k)^2 - m^2 &= -2pk \\ (p'+k)^2 - m^2 &= 2p'k \end{aligned}\right\} \quad p^2 = p'^2 = m^2, \; k^2 = 0$$

5. Then

$$i\mathcal{M} = \underbrace{\bar{u}(p')\mathcal{M}_0(p', p)\, u(p)}_{\text{elastic scattering}} \cdot \underbrace{\left[e\left(\frac{p' \cdot \varepsilon^*}{p' \cdot k} - \frac{p\,\varepsilon^*}{p\,k} \right)\right]}_{\text{bremsstrahlung}}$$

6. Scattering cross section (cf. (4.11)):

$$d\sigma(p \to p' + \gamma) = d\sigma(p \to p') \cdot \underbrace{\int \frac{d^3k}{(2\pi)^3} \sum_r \frac{e^2}{2|\vec{k}|} \left| \frac{p'\varepsilon^r}{p'k} - \frac{p\varepsilon^r}{pk} \right|^2}_{\equiv dP_k(p \to p')}$$

7. <u>Evaluation:</u>

$$\int dP_k = \frac{\alpha}{\pi} \int_0^\infty dk \, \frac{1}{k} \overbrace{\int \frac{d\Omega_k}{4\pi} \sum_r \left| \frac{p'\varepsilon^r}{p'\tilde{k}} - \frac{p\,\varepsilon^r}{p\tilde{k}} \right|^2}^{\equiv \mathcal{I}(p,p')}$$

$$\tilde{k} = \frac{k}{|\vec{k}|} = \begin{pmatrix} 1 \\ \hat{k} \end{pmatrix}$$

$$= \frac{\alpha}{\pi} \, \mathcal{I}(p,p') \cdot \Big[\underbrace{\log(\infty)}_{\text{Problem 1}} - \underbrace{\log(0)}_{\text{Problem 2}} \Big]$$

8. <u>Approximations:</u>

- Problem 1: Soft-photon approximation breaks down at $k \approx |\vec{q}| = |\vec{p} - \vec{p}'|$

  $\to$ Introduce upper cutoff at $|\vec{q}|$

- **Problem 2:** Probability of radiating a very soft photon is infinite!

→ IR-divergences of perturbative QED

Solution: Regularization with finite photon mass $\mu > 0$:

$$\frac{1}{k} = \frac{1}{E_k} \mapsto \frac{1}{\sqrt{\mu^2 + k^2}}$$

and therefore

asymptotics for $\mu \to 0$

$$\int_0^{|\vec{q}|} dk \, \frac{1}{\sqrt{\mu^2 + k^2}} = \log\left(\frac{\sqrt{\mu^2 + |\vec{q}|^2} + |\vec{q}|}{\mu}\right) \sim \log\left(2\frac{|\vec{q}|}{\mu}\right) \sim \log\left(\frac{|\vec{q}|}{\mu}\right) = \frac{1}{2}\log\left(\frac{|\vec{q}|^2}{\mu^2}\right)$$

- Relativistic limit ($E_{p,p'} \gg m$):

$$I(p,p') \simeq 2\log\left(\frac{-q^2}{m}\right) \qquad \text{with} \quad -q^2 = -(p-p')^2 \geq 0$$

Proof: ⇒ PES pp. 180–182, starting at Eq. (6.12)

9. Result:

$$\boxed{d\sigma(p \to p' + \gamma) \approx d\sigma(p \to p') \cdot \frac{\alpha}{\pi} \underbrace{\log\left(\frac{-q^2}{\mu^2}\right) \cdot \log\left(\frac{-q^2}{m^2}\right)}_{\text{Sudakov double logarithm}}}$$

for $\mu \to 0$ (regularization) and $E_{p,p'} \gg m$ (or $-q^2 = |\vec{q}|^2 \to \infty$)

10. Two problems:

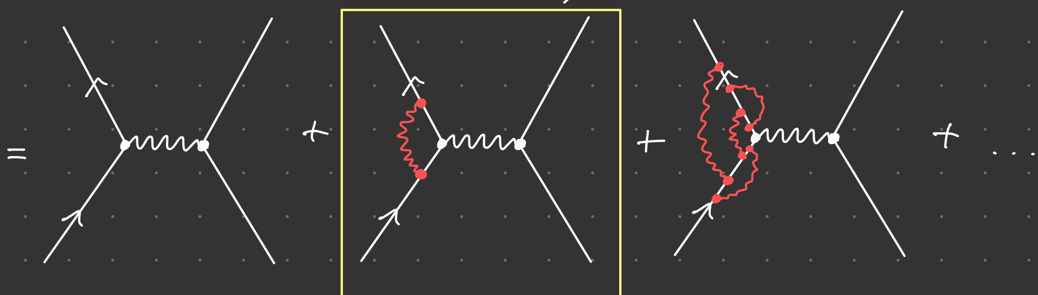
- Dependence of unphysical photon mass μ
- Logarithmic divergence for $-q^2 \to \infty$ ($\rightarrow$ cannot be interpreted as probability)

6.3. The Electron Vertex Function

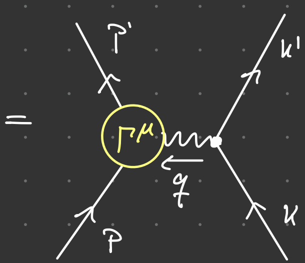
6.3.1. Formal Structure

1. Scattering amplitude:

$$i\mathcal{M}\left(e^-(p)\,\mu^-(k)\to e^-(p')\,\mu^-(k')\right)$$



$\Rightarrow$ next lecture



$$= ie^2\,\bar{u}_e(p')\,\Gamma^\mu(p',p)\,u_e(p)\,\frac{1}{q^2}\,\bar{u}_\mu(k')\,\gamma_\mu\,u_\mu(k)$$

2. General form:

$$\Gamma^\mu(p',p) = f\left(p^\mu, p'^\mu, \gamma^\mu, m, e, \mathbb{C}\right)$$

3. Restrictions

a) Lorentz covariance: Γ^μ transforms like $\gamma^\mu \longrightarrow$

$$\Gamma^\mu = A\cdot\gamma^\mu + \tilde{B}\cdot p^\mu + \tilde{C}\cdot p'^\mu$$

$$= A\cdot\gamma^\mu + B\cdot(p'^\mu + p^\mu) + C\cdot(p'^\mu - p^\mu) \qquad (6.1)$$

b) Recall $\not{p}\,u(p) = m\cdot u(p)$ and $\bar{u}(p')\not{p}' = \bar{u}(p')\cdot m \longrightarrow$

$$X = X(p^\mu, p'^\mu, m, e, \mathbb{C})\cdot\mathbb{1} \quad \text{for} \quad X = A, B, C$$

$q^2 = q'^2 = m^2$

c) $q^2 = (p' - p)^2 = 2(m^2 - p\cdot p')$ only non-trivial scalar $\longrightarrow$

$$X = X(q^2, m, e, \mathbb{C}) \quad \text{for } X = A, B, C$$

d) Ward identity for $U(1)$ gauge-symmetry of QED Lagrangian:

$$q_\mu \Gamma^\mu(p';p) \overset{*}{=} 0$$

(=) P&S pp. 238-244 for a proof and pp. 159-161 for a motivation

$$\rightarrow \quad 0 = q_\mu \Gamma^\mu = A \cdot \underbrace{q_\mu \gamma^\mu}_{=0} + B \cdot \underbrace{q_\mu (p'^\mu + p^\mu)}_{=0} + C \cdot q^2$$

$$\bar{u}(p')(\not{p}' - \not{p})\,\bar{u}(p)$$
$$= (m - m)\,\bar{u}(p')\bar{u}(p) = 0$$

$$\rightarrow \quad C = 0$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

4. Gordon identity

$$\bar{u}(p')\,\frac{p'^\mu + p^\mu}{2m}\,u(p) \overset{\circ}{=} \bar{u}(p')\gamma^\mu u(p) - \bar{u}(p')\,\frac{i\sigma^{\mu\nu}q_\nu}{2m}\,u(p) \qquad (6.2)$$

5. Therefore

$$\boxed{\Gamma^\mu(p',p) = \gamma^\mu \underbrace{F_1(q^2)}_{=1+\mathcal{O}(\alpha)} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\cdot \underbrace{F_2(q^2)}_{=0+\mathcal{O}(\alpha)}}$$

$F_i(q^2)$: form factors