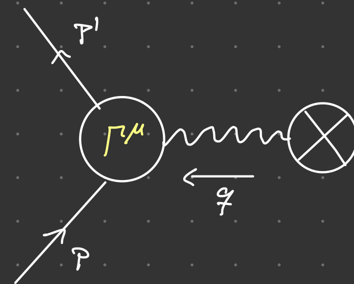
6.3.2. The Landé g-factor

1. Setting:

$\sphericalangle$ Classical, external field $A_\mu^{cl}(x)$: (⊜ Problemset 8)

$$H_{int} = e \int d^3x \, \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu^{cl}(x)$$

$\rightarrow \quad i\mathcal{M} (2\pi) \delta(p'^0 - p^0) =$



$A_\mu^{cl}(q)$

P-Set 8

$$\overset{\downarrow}{=} \; -ie \, \bar{u}(p') \Gamma^\mu(p',p) u(p) \cdot A_\mu^{cl}(q = p'-p)$$

2. Electric charge

a) $\sphericalangle$ $A_\mu^{cl}(x) = (\phi(x), \vec{0}) \implies A_\mu^{cl}(q) = (2\pi \delta(q^0) \phi(\vec{q}), \vec{0})$

b) $i\mathcal{M} = -ie \, \bar{u}(p') \Gamma^0(p',p) u(p) \cdot \phi(\vec{q})$

c) $\measuredangle\ \phi(\vec{x})$ slowly varying $\rightarrow$ $\phi(\vec{q})$ concentrated at $\vec{q}=0$ $\longrightarrow$ take limit $\vec{q}\rightarrow 0$:

$$i\mathcal{M} \approx -ie\, F_1(0)\, \bar{u}(p')\gamma^0 u(p) \cdot \phi(\vec{q}) \underset{|\vec{p}|^2 \ll m^2}{\approx} -ie\, F_1(0)\, \phi(\vec{q}) \cdot 2m\, \xi'^{\dagger}\xi$$

see (4.17)

d) $\xrightarrow{*}$ Born approximation with potential (recall (4.19))

$$V(\vec{x}) = \underbrace{e\, F_1(0)}_{\overset{!}{=}\, e}\, \phi(\vec{x})$$

e) Charge $e \overset{!}{=} e\, F_1(0)$ and $F_1^{(0)} = 1$ $\longrightarrow$

$$\boxed{F_1^{(n)}(0) = 0 \quad \text{for} \quad n \geq 1}$$

$$F_1 = \sum_{n=0}^{\infty} F_1^{(n)} \alpha^n$$

3. Magnetic moment:

a) $\measuredangle\ A_\mu^{cl}(x) = (0, \vec{A}(\vec{x})) \Rightarrow A_\mu^{cl}(q) = (0, 2\pi\delta(q^0)\vec{A}(\vec{q}))$

b) The

$$i\mathcal{M} = ie\,\bar u(p')\underbrace{\left[\gamma^i F_1(q^2) + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right]}_{\text{Vanishes for } q=0 \text{ and } |\vec p|^2 \ll m^2 \text{ of (4.17)}} u(p)\cdot A_{cl}^i(\vec q)$$

c) X F_1-term and expand bispinors in <u>linear</u> of $\vec p$ and $\vec p\,'$:

$$\bar u(p')\gamma^i u(p) \overset{\circ}{\approx} 2m\,\xi'^\dagger\left(\frac{\vec p\,'\cdot\vec\sigma}{2m}\cdot\sigma^i + \sigma^i\,\frac{\vec p\,\vec\sigma}{2m}\right)\xi$$

$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$

$$\overset{\circ}{=} \underbrace{\frac{p^i+p'^i}{2m}\cdot 2m\,\xi'^\dagger\xi}_{A} + \underbrace{2m\,\xi'^\dagger\left(\frac{-i}{2m}\,\varepsilon^{ijk}q^j\sigma^k\right)\xi}_{B}$$

$\vec p\cdot\vec A + \vec A\cdot\vec p$

→ Only (B) is spin-dependent and affects the magnetic moment!

d) $\times$ F_2-term and expand bispinors in lowest order of $\vec{p}$ and $\vec{p}'$:

$$\frac{i q_\nu}{2m} \bar{u}(p') \sigma^{i0} u(p) \overset{\circ}{\approx} 2m\, \xi'^{+} \left(\frac{-i}{2m} \varepsilon^{ijk} q^j \sigma^k \right) \xi$$

$u(p) \approx \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$ (4.17) and $[\sigma^i, \sigma^j] = 2i \varepsilon^{ijk} \sigma^k$ and $q_i = -q^i$

e) In summary:

$$iM \overset{q \to 0}{\approx} -ie\, \xi'^{+} \left\{ \frac{-1}{2m} \sigma^k \left[F_1(0) + F_2(0) \right] \right\} \xi \cdot \underbrace{\left[-i \varepsilon^{ijk} q^j A^i_{cl}(\vec{q}) \right]}_{= B^k_{cl}(\vec{q})} \cdot (2m)$$

with $\vec{B}_{cl} = \nabla \times \vec{A}_{cl}$

f) $\xrightarrow{*}$ Born approximation with potential

$$V(\vec{x}) = - \langle \vec{\mu} \rangle \cdot \vec{B}_{cl}(\vec{x})$$

yields the magnetic moment

$$\langle \vec{\mu} \rangle = \frac{e}{m}\left[F_1(0) + F_2(0)\right] \cdot \overbrace{\xi^{\dagger} \frac{\vec{\sigma}}{2} \xi}^{\langle \vec{S} \rangle} \equiv g \cdot \mu_B \cdot \langle \vec{S} \rangle$$

with Bohr magneton $\mu_B = \dfrac{e}{2m}$ and

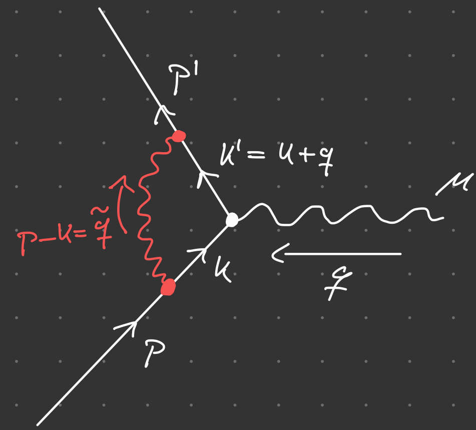
$\alpha F_2^{(1)}(0) + \mathcal{O}(\alpha^2)$

$$\boxed{\begin{aligned} g &= 2\Big[\overbrace{F_1(0)}^{1} + F_2(0)\Big] = 2 + 2F_2(0) \\ &= \underbrace{2}_{\text{Dirac equation}} + \underbrace{2\alpha F_2^{(1)}(0) + \mathcal{O}(\alpha^2)}_{\text{Anomalous magnetic moment}} \end{aligned}}$$

6.3.3. Evaluation

1. Scattering amplitude:

$$\bar{u}(p')\left[e\Gamma^{(1)}(p',p)\right]^{\mu} u(p) \quad =$$



$$= \int \frac{d^4k}{(2\pi)^4} \; \frac{-i g_{\nu\rho}}{\tilde{q}^2 + i\varepsilon} \, \bar{u}(p')(-ie\gamma^{\nu}) \, \frac{i(\not{k}'+m)}{k'^2 - m^2 + i\varepsilon} \, \gamma^{\mu} \, \frac{i(\not{k}+m)}{k^2 - m^2 + i\varepsilon} \, (-ie\gamma^{\rho}) \, u(p)$$

Contraction identities: $\gamma^{\nu}\gamma^{\mu}\gamma_{\nu} = -2\gamma^{\mu}$ etc.

$$\overset{\circ}{=} 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\left[\not{k}\gamma^{\mu}\not{k}' + m^2\gamma^{\mu} - 2m(k+k')^{\mu}\right] u(p)}{(\tilde{q}^2 + i\varepsilon)(k'^2 - m^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)} \qquad (6.3)$$

2. Feynman parameters:

$$\frac{1}{A_1 \ldots A_n} = \prod_{i=1}^{n} \int_0^1 dx_i \; \delta\Big(\sum_i x_i - 1\Big) \frac{(n-1)!}{[x_1 A_1 + \ldots + x_n A_n]^n}$$

Proof: $\Rightarrow$ Problemset 9 $\qquad$ x_i: Feynman parameters

3. Application to denominator of (6.3):

$$\frac{1}{(\tilde{q}^2 + i\varepsilon)(k'^2 - m^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)} = \int_0^1 dx\,dy\,dz\; \delta(x+y+z-1) \frac{2}{D^3}$$

with $\qquad x+y+z=1\,, \quad \tilde{q} = p-k\,, \quad k' = k+q$

$$D \overset{\circ}{=} k^2 + 2k\,(yq - zp) + y q^2 + z p^2 - (x+y)\, m^2 + i\varepsilon$$

Complete the square: $\quad l \equiv k + yq - zp$

$$= l^2 - \Delta + i\varepsilon$$

where $\quad \Delta \equiv -xy\,q^2 + (1-z)^2 m^2 > 0 \quad$ (since $q^2 < 0$)

4. Express the numerator of (6.3) in terms of l ($k^\mu = l^\mu - y q^\mu + z p^\mu$):

$$\bar u(p')\left[\not k \gamma^\mu \not k' + m^2 \gamma^\mu - 2m (k+k')^\mu\right] u(p)$$

- $\displaystyle\int \frac{d^4 l}{(2\pi)^4} \frac{l^\mu}{D(l^2)} = 0$
- $\displaystyle\int \frac{d^4 l}{(2\pi)^4} \frac{l^\mu l^\nu}{D(l^2)} = \int \frac{d^4 l}{(2\pi)^4} \cdot \frac{g^{\mu\nu}}{4} \cdot \frac{l^2}{D(l^2)}$

$$\hat{=}\ \bar u(p') \left\{ \begin{array}{l} -\frac{1}{2}\gamma^\mu l^2 + [-y \not q + z \not p]\, \gamma^\mu\, [(1-y)\not q + z \not p] \\ + m^2 \gamma^\mu - 2m\left[(1-2y) q^\mu + 2 z p^\mu\right] \end{array} \right\} u(p)$$

under the $\int d^4 l$ integral

- $\not p \gamma^\mu = 2p^\mu - \gamma^\mu \not p$
- $\bar u(p') \not q\, u(p) = 0$
- $x + y + z = 1$

Structure expected, see (6.1)

$$\hat{=}\ \bar u(p') \left\{ \gamma^\mu \underbrace{\left[-\frac{1}{2} l^2 + (1-x)(1-y) q^2 + (1-2z-z^2) m^2\right]}_{\hat{=}\, A} + (p'+p)^\mu \underbrace{\left[m z (z-1)\right]}_{\hat{=}\, B} + q^\mu \underbrace{\left[m (z-2)(x-y)\right]}_{\hat{=}\, C} \right\} u(p)$$

5. C is antisymmetric and D is symmetric under $x \leftrightarrow y$ $\rightarrow$ drop C

6. Gordon identity (6.2) $\longrightarrow$

$$\bar{u}(p')[\delta\Gamma^{(1)}(p';p)]^\mu u(p) = 2ie^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx\,dy\,dz\, \delta(x+y+z-1) \frac{2}{[D(l^2)]^3}$$

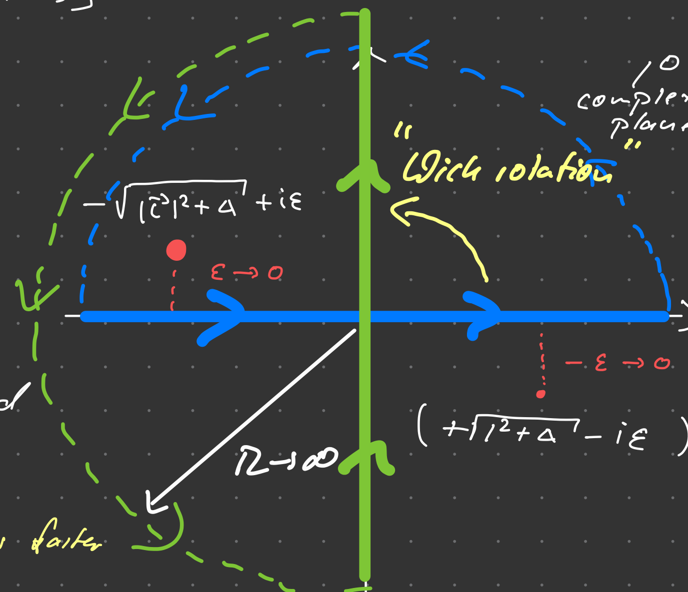
$$\times\ \bar{u}(p') \left\{ \gamma^\mu \left[-\frac{1}{2} l^2 + (1-x)(1-y)\, q^2 + (1-4z+z^2)\, m^2 \right] + \frac{i\sigma^{\mu\nu}}{2m} \cdot \left[2m^2 z(1-z) \right] \right\} u(p)$$

7. Momentum integral:

a) Problem: $l^2 = (l^0)^2 - \vec{l}^{\,2}$

Solution: Wick rotation

= Evaluation of a contour integral along a rotated contour that encircles the same poles:



(Integrand vanishes faster than $\frac{1}{|l^0|}$)

Parametrization of the new contour:

$$\boxed{\begin{aligned} &l^0 \equiv i \cdot l_E^0 \quad \text{and} \quad \vec{l} \equiv \vec{l}_E \qquad \text{with } l_E \in \mathbb{R}^4 \\ &\Rightarrow \underset{\uparrow \text{ Minkowski norm}}{l^2} = l^{0\,2} - \vec{l}^{\,2} = -(l_E^0)^2 - \vec{l}_E^{\,2} = \underset{\uparrow \text{ Euclidean norm}}{-\,l_E^2} \end{aligned}}$$

b) The $(m>2)$

Wick rotation

$$\lim_{\varepsilon\to 0} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2-\Delta+i\varepsilon)^m} = \frac{i}{(-1)^m} \frac{1}{(2\pi)^4} \int d^4 l_E \frac{1}{(l_E^2+\Delta)^m}$$

$$= \frac{i(-1)^m}{(2\pi)^4} \underbrace{\int d\Omega_4}_{=2\pi^2} \underbrace{\int_0^\infty d l_E \frac{l_E^3}{(l_E^2+\Delta)^m}}_{= \frac{1}{2(m-1)(m-2)\Delta^{m-2}}}$$

$$= \frac{i(-1)^m}{(4\pi^2)} \frac{1}{(m-1)(m-2)\,\Delta^{m-2}} \qquad (6.5)$$

and similarly ($m>3$)

$$\lim_{\varepsilon\to 0}\int\frac{d^4l}{(2\pi)^4}\frac{l^2}{(l^2-\Delta+i\varepsilon)^m} \overset{\circ}{=} \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)\,\Delta^{m-3}} \qquad (6.6)$$

<u>Problem:</u> For $m=3$ the integral diverges! (UV-divergence for $l_E \sim \vec{k}^2 \to \infty$)

c) Fix: Pauli-Villars regularization:

$$\frac{-ig_{\mu\nu}}{\tilde{q}^2+i\varepsilon} \longmapsto \frac{-ig_{\mu\nu}}{\tilde{q}^2+i\varepsilon} - \frac{-ig_{\mu\nu}}{\tilde{q}^2-\Lambda^2+i\varepsilon}$$

for large Λ

<u>Hope:</u> Λ does not appear in physical predictions

$\overset{\circ}{\longrightarrow}$ Only change: $\Delta_\Lambda = -xyq^2 + (1-z)^2 m^2 + z\Lambda^2$

d) Therefore $(n=3)$

- $(6.5) \mapsto (6.5) - \mathcal{O}(\Delta_\Lambda^{-1}) = (6.5) - \mathcal{O}(\Lambda^{-2}) \approx (6.5)$
- $(6.6) \mapsto \lim_{\varepsilon \to 0} \int \frac{d^4 l}{(2\pi)^4} \left[\frac{l^2}{(l^2-\Delta+i\varepsilon)^3} - \frac{l^2}{(l^2-\Delta_\Lambda+i\varepsilon)^2} \right]$

Wick rotation $\Leftrightarrow$ Problemset 11

$$\hat{=} \frac{i}{(4\pi)^2} \log\left(\frac{\Delta_\Lambda}{\Delta}\right) \xrightarrow{\Lambda \to \infty} \frac{i}{(4\pi)^2} \log\left(\frac{z\Lambda^2}{\Delta}\right)$$

8. Result:

$$\bar{u}(p')\left[\alpha\Gamma^{(1)}(p',p)\right]^\mu u(p) \hat{=} \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$

$$\times\, \bar{u}(p') \left\{ \gamma^\mu \overbrace{\left[\log\left(\frac{z\Lambda^2}{\Delta}\right) + \frac{(1-x)(1-y)q^2}{\Delta} + \frac{(1-4z+z^2)m^2}{\Delta} \right]}^{\hat{=} F_1^{(1)}(q^2)} + \frac{i\sigma^{\mu\nu}q_\nu}{2m} \underbrace{\left[\frac{2m^2 z(1-z)}{\Delta} \right]}_{\hat{=} F_2^{(1)}(q^2)} \right\} u(p)$$

9. Discussion of F_1:

a) Problem 1: It should be $F_1^{(1)}(0)=0$, but here $F_1^{(1)}(0)\neq 0$!

Fix 1: $$F_1^{(1)}(q^2) \longmapsto F_1^{(1)}(q^2) - F_1^{(1)}(0)$$

b) Problem 2: In addition, there is a IR-divergence (for $\tilde{q}^2 \to 0$)

For $q^2=0$ $\rightarrow$ $$\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\, \frac{1-4z+z^2}{(1-z)^2}$$

$$= \int_0^1 dz \int_0^{1-z} dy\, \frac{-2+(1-z)(3-z)}{(1-z)^2} = \int_0^1 dz\, \frac{-2}{1-z} + \text{finite terms}$$

Fix 2: Add a small photon mass $\mu>0$ $\longrightarrow$

$$\Delta \longmapsto \Delta_\mu = -xyq^2 + (1-z)^2 m^2 + z\mu^2$$

c) Fix 1 + Fix 2 $\longrightarrow$ $F_1(q^2) = 1 + \alpha F_1^{(1)}(q^2) + \mathcal{O}(\alpha^2)$ with

$$\boxed{F_1^{(1)}(q^2) = \frac{1}{2\pi} \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1) \times \left[\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2 xy} \right) + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2 xy + z\mu^2} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + z\mu^2} \right]}$$

10. Discussion of F_2: $F_2(q^2) = \alpha F_2^{(1)}(q^2) + \mathcal{O}(\alpha^2)$ with

$$\boxed{F_2^{(1)}(q^2) = \frac{1}{2\pi} \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1) \left[\frac{2m^2 z(1-z)}{m^2(1-z)^2 - q^2 xy} \right]}$$

11. Landé g-factor:

$$F_2(q^2=0) = \frac{\alpha}{2\pi}\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\delta(x+y+z-1)\,\frac{2z}{1-z} + \mathcal{O}(\alpha^2)$$

$$= \frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z} dy\,\frac{2z}{1-z} + \mathcal{O}(\alpha^2)$$

$$= \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2)$$

Therefore the anomalous magnetic dipole moment of the electron is

$$\boxed{a_e \equiv \frac{g-2}{2} = \frac{\alpha}{2\pi} \approx 0.0011614 \qquad a_e^{exp} \approx 0.0011597}$$

$\Big\}\ \alpha^2 \sim 0.5 \cdot 10^{-4}$

Note 6.1

- Our first-order result was obtained by Schwinger in 1948
- Modern values:

$$a_e^{QED} = 0.001\,159\,652\,181\,643\,(763)$$

$$a_e^{exp} = 0.001\,159\,652\,180\,73\,(28)$$

$\rightarrow$ Agree to 11 significant digits

- First order also valid for muons: $a_\mu^{(1)} = \frac{\alpha}{2\pi} = a_e^{(1)}$

$$a_\mu^{exp} - a_\mu^{SM} = 261\,(63)(48) \times 10^{-11}$$