## 6.3.4. The Infrared Divergence

1. Goal: Understand asymptotics of $|F_1(q^2)| \longrightarrow \infty$ for $\mu \rightarrow 0$

2. Show

$\Rightarrow$ P-Set 10

$$F_1(q^2) = (6.11) \overset{\mu \rightarrow 0}{\sim} 1 - \frac{\alpha}{2\pi} f_{IR}(q^2) \cdot \log\left(\frac{A}{\mu^2}\right) + \mathcal{O}(\alpha^2) \qquad (6.12)$$

where $A \in \{-q^2, m^2\}$ and

$$f_{IR}(q^2) = \int_0^1 d\xi \, \frac{m^2 - q^2/2}{m^2 - q^2 \xi(1-\xi)} - 1 \geq 0$$

3. $\Rightarrow$ Cross section for electron scattering off a static potential:

$$\frac{d\sigma(\vec{p} \rightarrow \vec{p}\,')}{d\Omega} \sim \underbrace{\left(\frac{d\sigma}{d\Omega}\right)_0}_{\text{Tree-level result}} \times \left[ 1 \underbrace{- \frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{A}{\mu^2}\right)}_{\text{Problem: } \rightarrow -\infty \text{ for } \mu \rightarrow 0} + \mathcal{O}(\alpha^2) \right]$$

4. Limit $-q^2 \to \infty$:

$$f_{IR}(q^2) \sim \int_0^1 d\xi \, \frac{-q^2/2}{-q^2\,\xi(1-\xi)} \overset{\circ}{\sim} \log\left(\frac{-q^2}{m^2}\right)$$

$$\to \quad F_1(-q^2 \to \infty) \overset{\mu \to 0}{\sim} 1 - \frac{\alpha}{2\pi} \underbrace{\log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right)}_{\text{Sudakov double logarithm}} + \mathcal{O}(\alpha^2)$$

5. Comparison with bremsstrahlung (6.3) for $-q^2 \to \infty$:

$$\frac{d\sigma(\vec{p} \to \vec{p}\,')}{d\Omega} \overset{\mu \to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^4)\right]$$

$$\frac{d\sigma(\vec{p} \to \vec{p}\,' + \gamma)}{d\Omega} \overset{\mu \to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[+ \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]$$

$\to$ Both divergent but their sum is finite and independent of $\mu$!

6. Suggested solution: Photon detectors cannot detect photons below a lower threshold $E_{min}$:

$$\rightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{measure} = \frac{d\sigma(\vec{p}\to\vec{p}\,')}{d\Omega} + \frac{d\sigma\left(\vec{p}\to\vec{p}\,'+\gamma\ (k<E_{min})\right)}{d\Omega}$$

7. For general $q$:

$$\left(\frac{d\sigma}{d\Omega}\right)_{measured} \overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \Bigg[\underbrace{1-\frac{\alpha}{\pi} f_{IR}(q^2)\cdot\log\left(\frac{\Lambda}{\mu^2}\right)}_{\text{Elastic scattering}} + \underbrace{\frac{\alpha}{2\pi}\mathcal{I}(p,p')\log\left(\frac{E_{min}^2}{\mu^2}\right)}_{\text{Bremsstrahlung}} + O(\alpha^2)\Bigg]$$

with $\mathcal{I}(p,p')$ defined in (6.1)

$$d\sigma(p\to p'+\gamma) = d(p\to p')\cdot\frac{\alpha}{2\pi}\mathcal{I}(p,p')\cdot\log\left(\frac{|\vec{q}|^2}{\mu^2}\right)$$

$E_{min}^2$

8. Show (using a Feynman parameter)

$$\gamma(p,p') \overset{*}{=} 2 \cdot f_{IR}(q^2) \quad \text{for all } p,p' \qquad (6.13)$$

Proof: ⇒ PES p. 201

9. Then

$$\boxed{\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} &\overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \cdot \left[1 - \frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{A}{E_{min}^2}\right) + O(\alpha^2)\right] \\ &\overset{-q^2 \gg m^2}{\approx} \left(\frac{d\sigma}{d\Omega}\right)_0 \cdot \left[1 - \underbrace{\frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{E_{min}^2}\right)} + O(\alpha^2)\right] \qquad (6.14) \end{aligned}}$$

Correction by Sudakov double logarithm

→ Independent of $\mu$ but dependent on experimental conditions ($E_{min}$) (fine!)

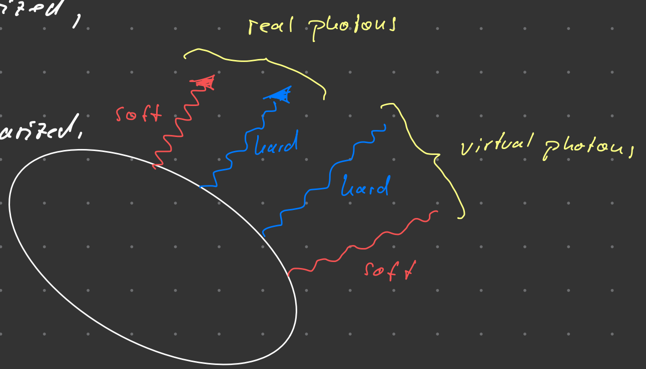
## 6.3.5 Summation and Interpretation of Infrared Divergences

### 1. Problems:

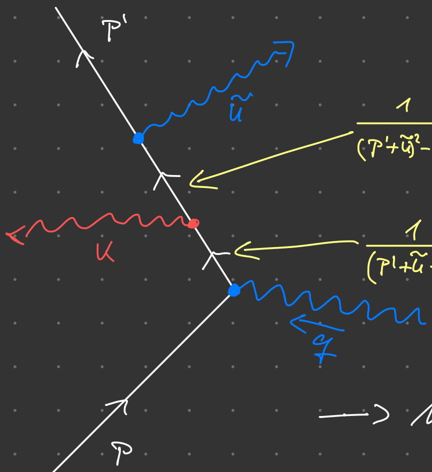
- Did not show cancellation of IR-divergences for higher orders
- Cross section (6.14) becomes negative for $E_{min} \to 0$

### 2. Notation:

- Real photons are on-shell, transversely polarized, connected with only one end to the diagram
- Virtual photons can be off-shell, longitudinally polarized, connected with both ends to the diagram
- Soft photons: momentum is upper-bounded: $k_E^2 < E_{min}^2$ (virtual) and $|\vec{k}| < E_{min}$ (real)
- Hard photons: momentum is lower-bounded: $k_E^2 > E_{min}^2$ (virtual) and $|\vec{k}| > E_{min}$ (real)



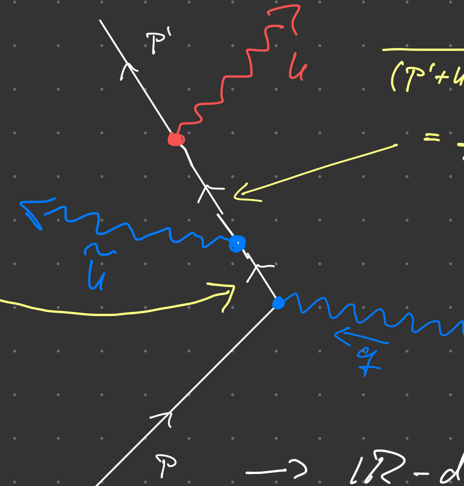
## 3. Origin of IR divergences:




$$\frac{1}{(p'+\tilde{k})^2-m^2} = \frac{1}{2p'\underbrace{\tilde{k}}_{\not\to 0}} < \infty$$

$$\frac{1}{(p'+\tilde{k}+k)^2-m^2} = \frac{1}{2p'\underbrace{(k+\tilde{k})}_{\not\to 0} + 2\underbrace{k\tilde{k}}_{\to 0}} < \infty$$

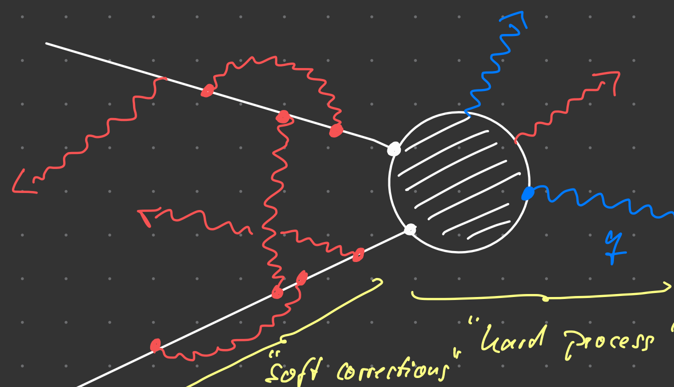
⟶ No IR-divergence



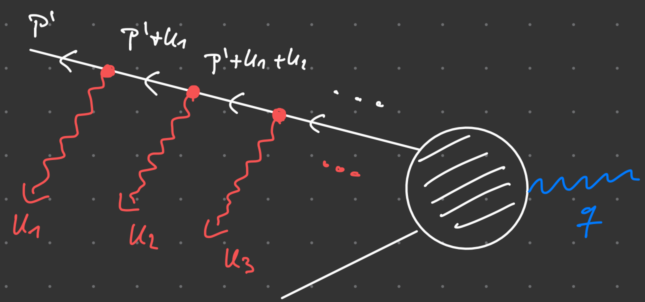

$$\frac{1}{(p'+k)^2-m^2} = \frac{1}{2p'\underbrace{k}_{\to 0}} \longrightarrow \infty$$

⟶ IR-divergence

## 4. → Generic process



5. $\gamma$ Outgoing leg:




a) Feynman rules $\longrightarrow$

$$\bar{u}(p')\,(-ie\gamma^{\mu_1})\,\frac{i(\not{p}'+\cancel{\not{k}_1}+m)}{2p'\cdot k_1+\cancel{O(k^2)}}\,(-ie\gamma^{\mu_2})\,\frac{i(\not{p}'+\cancel{\not{k}_1}+\cancel{\not{k}_2}+m)}{2p'\cdot(k_1+k_2)+\cancel{O(k^2)}}\times\ldots\ldots$$

$$-ie\gamma^{\mu_n}\,\frac{i(\not{p}'+\cancel{\not{k}_1}+\ldots+\cancel{\not{k}_n}+m)}{2p'\cdot(k_1+\ldots+k_n)+\cancel{O(k^2)}}\,(i\mathcal{M}_{hard})\ldots$$

b) Soft-photon approximation $(k_i \to 0)$

- Drop non-singular terms $\cancel{\not{k}_i}$
- Drop $O(k^2)$ terms
- $\gamma^\mu(\not{p}'+m) = (-\not{p}'+m)\gamma^\mu + 2p'^\mu$
- $\bar{u}(p')(-\not{p}'+m) = 0$

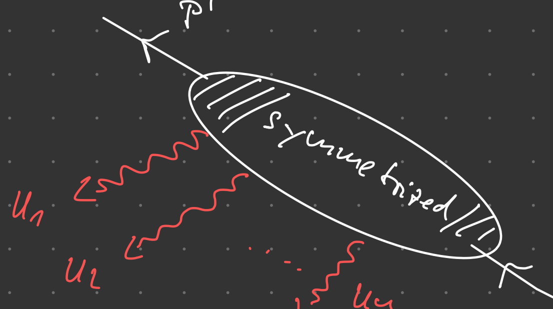
$$\overset{\circ}{\longrightarrow} \quad \underbrace{\bar{u}(p') \left(e\frac{p'^{\mu_1}}{p'\cdot k_1}\right)\left(e\frac{p'^{\mu_2}}{p'\cdot(k_1+k_2)}\right) \cdots\cdots e\frac{p'^{\mu_n}}{p'\cdot(k_1+\dots+k_n)}}_{=\square} \cdots$$

c) Sum over all orderings of $k_1 \dots k_n$:

$$\sum_{\pi\in S_n} \square(k_i \to k_{\pi(i)}) = ?$$

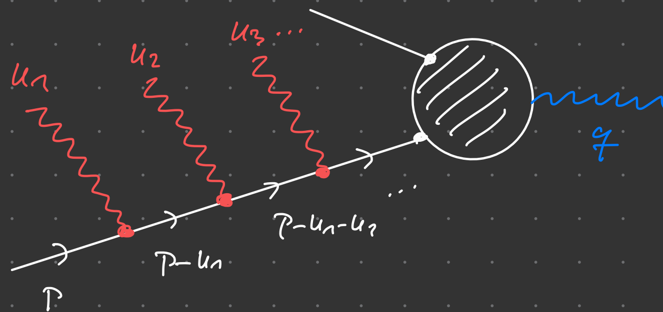
d) Use $$\sum_{\pi\in S_n} \frac{1}{p\cdot k_{\pi(1)}} \cdot \frac{1}{p\cdot(k_{\pi(1)}+k_{\pi(2)})} \cdots \frac{1}{p\cdot(k_{\pi(1)}+\dots+k_{\pi(n)})} \overset{\circ}{=} \prod_{i=1}^{n} \frac{1}{p\cdot k_i}$$

Proof by induction over $n$.

e) 


$$\approx \bar{u}(p') \cdot \prod_{i=1}^{n} \left(e\frac{p'^{\mu_i}}{p'\cdot k_i}\right) \qquad (6.15)$$

6. § Incoming leg:

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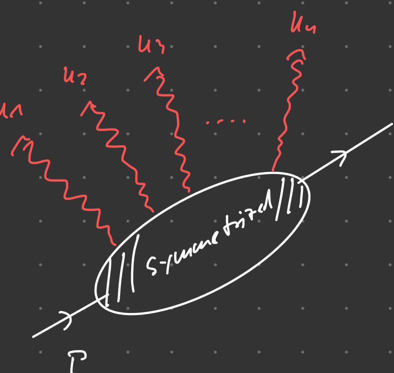
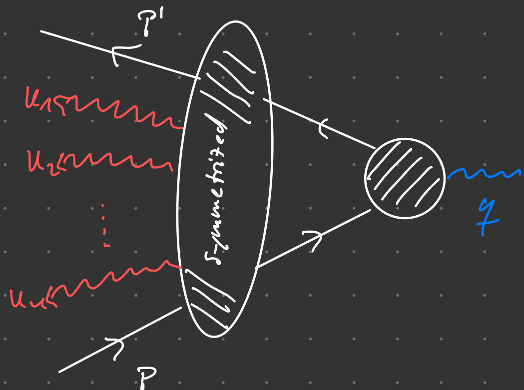



$$\approx \prod_i \left( -e \frac{p^{\mu_i}}{p \cdot k_i} \right) u(p) \qquad (6.16)$$

7. & Sum over $n$ soft photons attached either to the incoming or the outgoing leg:

$(6.15) + (6.16) \longrightarrow$



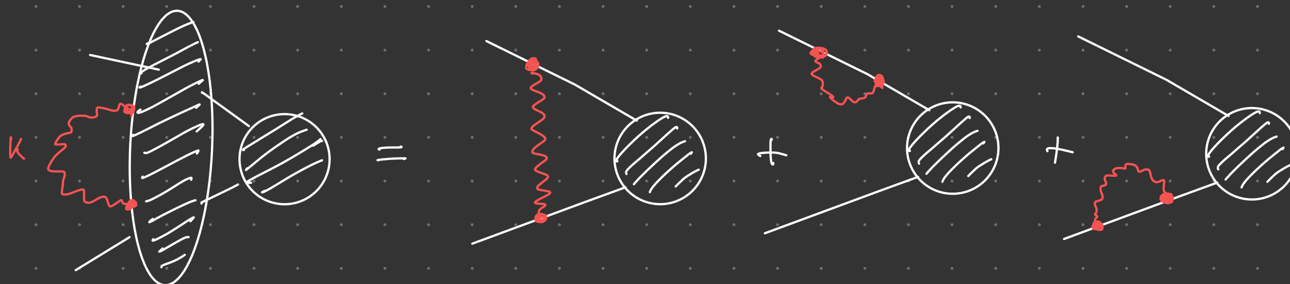

$$\simeq \bar{u}(p')\, i\mathcal{M}_{\text{hard}}\, u(p) \times \prod_i e\left(\frac{p'^{\mu_i}}{p'\cdot k_i} - \frac{p^{\mu_i}}{p\cdot k_i}\right) \qquad (6.17)$$

8. Virtual photon between vertex $i$ and $j$:

- Set $k_j = -k_i \equiv k$
- photon propagator $\dfrac{-i g_{\mu\nu}}{k^2 + i\varepsilon}$
- Integrate over $k$
- Multiply by $\frac{1}{2}$ to account for the symmetry $k_i \leftrightarrow k_j$

$$\longrightarrow \quad \frac{e^2}{2}\int \frac{d^4k}{(2\pi)^4}\cdot\frac{-i}{k^2+i\varepsilon}\left(\frac{p'^\mu}{p'\cdot k}-\frac{p^\mu}{p\cdot k}\right)\cdot\left(\frac{p'_\mu}{-p'\cdot k}-\frac{p_\mu}{-p\cdot k}\right) = X \qquad (6.18)$$

9. <u>Evaluation of $X$:</u> Special case with one virtual photon:

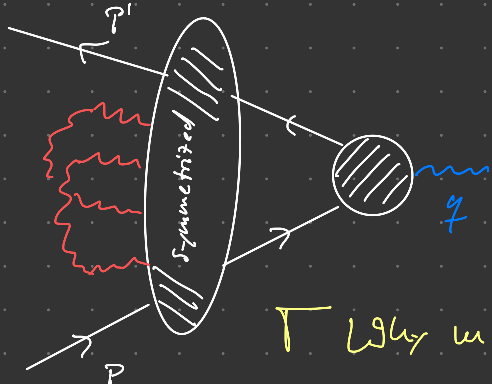



$$\approx \bar{u}(p')\, i\mathcal{M}_{\text{hard}}\, u(p) \times X$$

$$\stackrel{!}{=} \bar{u}(p')\, i\mathcal{M}_{\text{hard}}\, u(p) \times \Big[\underbrace{-\frac{\alpha}{2\pi}\, f_{IR}(q^2)\,\log\Big(\frac{-q^2}{\mu^2}\Big)}_{\text{IR-Limit of } F_1^{(1)}(q^2),\ \text{see } (6.12)}\Big]$$

$$\rightarrow \quad \boxed{\mathcal{X} = -\frac{\alpha}{2\pi} f_{IR}(q^2)\log\left(\frac{-q^2}{\mu^2}\right)} \qquad (6.19)$$

10. $\sum$ Sum of arbitrary many soft, virtual photons: (6.17) + (6.18) $\longrightarrow$




$$\approx \bar{u}(p')\, i\mathcal{M}_{hard}\, u(p) \times \sum_{m=0}^{\infty} \frac{\mathcal{X}^m}{m!} = \bar{u}(p')\, i\mathcal{M}_{hard}\, u(p)\cdot \exp(\mathcal{X}) \qquad (6.20)$$

[ Why $m!$ ?

equivalent $\rightarrow$ divide by $\frac{1}{m!} = \frac{1}{2}$

11. § Emission of a <u>real</u> photon $k_i = k$:

- multiply by $[\varepsilon_\mu^r(k)]^*$
- square the amplitude
- Phase-space integration of $\vec{k}$ up to $|\vec{k}| < E_{min}$
- Sum over polarisation $r = 1, 2$

$$\to \int^{E_{min}} \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2k} \sum_r e^2 \left| \frac{p' \cdot \varepsilon^r}{p' \cdot k} - \frac{p \cdot \varepsilon^r}{p \cdot k} \right|^2 \equiv Y$$

Recall (6.1) and (6.2)

$$\sim \frac{\alpha}{\pi} \mathcal{I}(p, p') \log\left(\frac{E_{min}}{\mu}\right) \overset{(6.13)}{=} \frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{E_{min}^2}{\mu^2}\right) \qquad (6.21)$$

12. § Cross section for emission of arbitrary number of soft photons:

$$\sum_{n=0}^{\infty} \frac{d\sigma}{d\Omega}(\vec{p} \to \vec{p}' + n\gamma) = \underbrace{\frac{d\sigma}{d\Omega}(\vec{p} \to \vec{p}')}_{\propto |\bar{u}(p')\, i\mathcal{M}_{hard}\, u(p)|^2} \times \sum_{n=0}^{\infty} \frac{1}{n!} Y^n = \frac{d\sigma}{d\Omega}(\vec{p} \to \vec{p}') \exp(Y) \qquad (6.22)$$

13 → Measured cross section for process

$$\boxed{e^-(\vec{p}) \longrightarrow e^-(\vec{p}\,') + (\text{Any number of photons with } |\vec{k}| < E_{min})}$$

to all orders of $\alpha$ is

$$\boxed{\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{measured} &\overset{(6.20)+(6.22)}{\approx} \left(\frac{d\sigma}{d\Omega}\right)_0 \times \exp\left(2\mathcal{X} + \mathcal{Y}\right) \\
&\overset{(6.19)+(6.21)}{=} \left(\frac{d\sigma}{d\Omega}\right)_0 \times \underbrace{\exp\left[-\frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{-q^2}{E_{min}^2}\right)\right]}_{\geq 0 \text{ and } \leq 1} \\
&\overset{-q^2 \gg m^2}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \times \underbrace{\exp\left[-\frac{\alpha}{2\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{E_{min}^2}\right)\right]}_{\text{Sudakov form factor}}
\end{aligned}}$$

## Note 6.2

- As the result is independent of $\mu$, it demonstrates the cancellation of IR divergences in all orders of $\alpha$
- We can recover our previous result (6.14) by expanding the exponential