6.4. Field - Strength Renormalization

6.4.1. Structure of Two-Point Correlators in Interacting Theories

Here: $\lambda\,\phi^4$-theory (later: QED)

1. Goal: Study structure of $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ in an interacting theory

2. Interpretation for free theory:

$\langle 0|T\phi(x)\phi(y)|0\rangle$ = Amplitude of particle to propagate from y to x (for $x^0 > y^0$)

$\rightarrow$ Effects of interactions?

3. Mathematical preliminaries:

a) $\exists$ Hilbert space of interacting theory $\mathcal{H}_{int}$

b) Basis of $\mathcal{H}_{int}$:

$[H, \vec{P}] = 0 \rightarrow |\lambda_{\vec{p}}\rangle$ eigenstates with $E_{\vec{p}}(\lambda)$ and momentum $\vec{p}$

$\uparrow$

include multi-particle states with total momentum $\vec{p}$

c) $\exists$ Boost $\Lambda_{\vec{p}} \in SO^{+}(1,3)$ such that

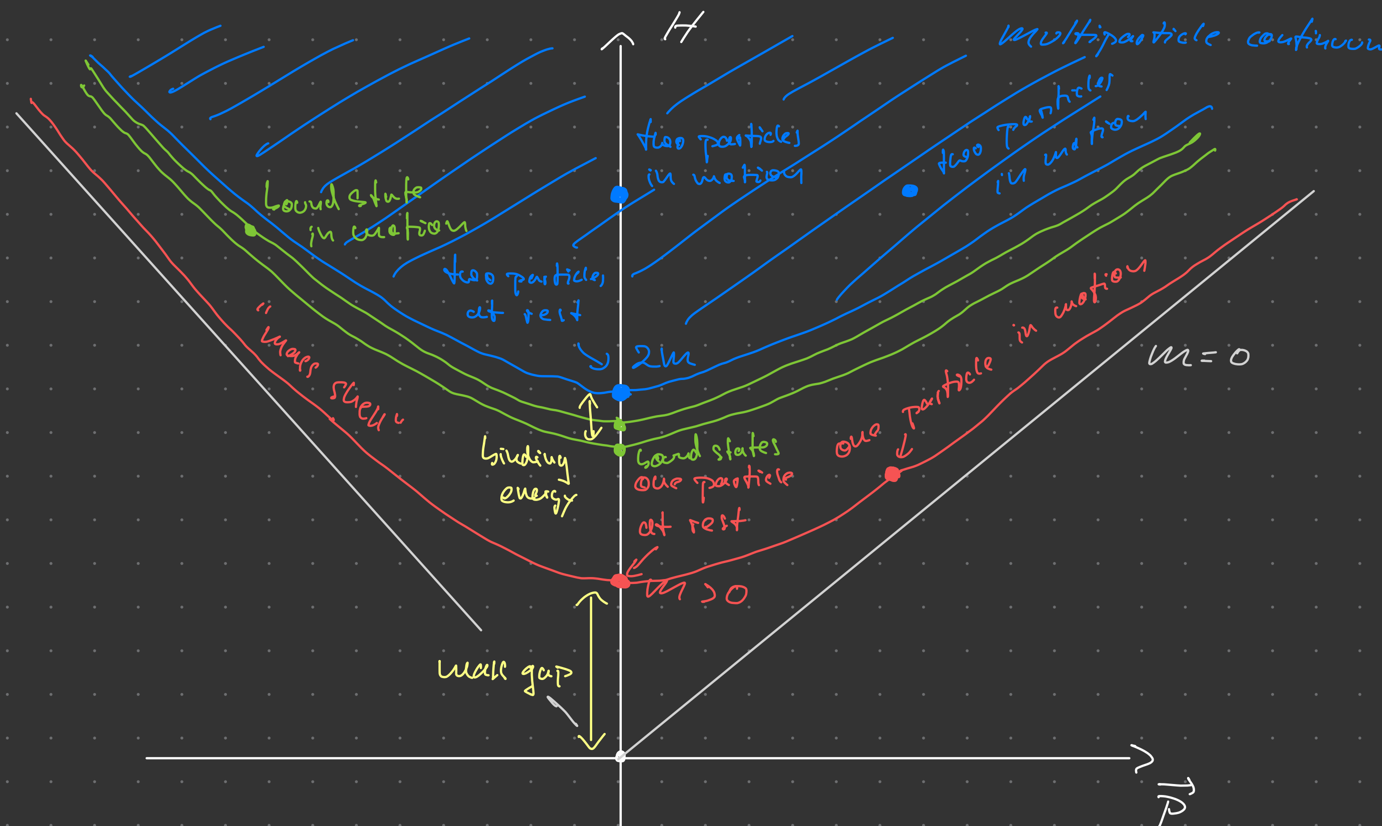
$$\Lambda_{\vec{p}} \begin{pmatrix} m_\lambda \\ \vec{0} \end{pmatrix} = \begin{pmatrix} E_{\vec{p}}(\lambda) \\ \vec{p} \end{pmatrix} \quad \text{with} \quad E_p(\lambda) = \sqrt{|\vec{p}|^2 + m_\lambda^2}$$

$\rightarrow \forall |\lambda_{\vec{p}}\rangle \; \exists \Lambda_{\vec{p}} \; \exists |\lambda_0\rangle : \quad |\lambda_{\vec{p}}\rangle = U(\Lambda_{\vec{p}})|\lambda_0\rangle$ with

$$H|\lambda_0\rangle = m_\lambda |\lambda_0\rangle \quad \text{und} \quad \vec{P}|\lambda_0\rangle = 0$$

$$H|\lambda_{\vec{p}}\rangle = E_{\vec{p}}(\lambda)|\lambda_{\vec{p}}\rangle \quad \text{und} \quad \vec{P}|\lambda_{\vec{p}}\rangle = \vec{p}|\lambda_{\vec{p}}\rangle$$

d) Typical spectrum of $P^\mu = (H, \vec{P})$ of an interacting theory with a mass gap:



e) Identity on $\mathcal{H}_{int}$:

Recall: $(\mathbb{1})_{1\text{-particle}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |\vec{p}\rangle\langle\vec{p}|$

$$\rightarrow \quad \mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}(\lambda)} |\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|$$

4. Insert identity $\rightarrow$ $(x^0 > y^0)$

drop

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}(\lambda)} \langle\Omega|\phi(x)|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|\phi(y)|\Omega\rangle + \text{const.}$$

$P \to p$, $p^0 = E_{\vec{p}}(\lambda)$

5. With $\langle\Omega|\phi(x)|\lambda_{\vec{p}}\rangle = \langle\Omega|e^{iPx}\phi(0)e^{-iPx}|\lambda_{\vec{p}}\rangle$

$$= \langle\Omega|\phi(0)|\lambda_{\vec{p}}\rangle e^{-ipx}\Big|_{p^0=E_{\vec{p}}(\lambda)}$$

$$= \langle\Omega|\overbrace{U(\Lambda_{\vec{p}})\underbrace{U^\dagger(\Lambda_{\vec{p}})}^{\mathbb{1}}\phi(0)U(\Lambda_{\vec{p}})}_{\phi(\Lambda_{\vec{p}}^{-1}0)=\phi(0)}|\lambda_0\rangle e^{-ipx}\Big|_{p^0=E_{\vec{p}}(\lambda)}$$

$$= \langle\Omega|\phi(0)|\lambda_0\rangle e^{-ipx}\Big|_{p^0=E_{\vec{p}}(\lambda)}$$

6. .... we find

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda{}' |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}(\lambda)} e^{-ip(x-y)} \Big|_{p^0 = E_{\vec p}(\lambda)}$$

Introduce p^0-integration (⊜ Eq. (2.5) ff)

$$\overset{x^0>y^0}{=} \sum_\lambda{}' |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \underbrace{\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m_\lambda+i\varepsilon} e^{-ip(x-y)}}_{\equiv D_F(x-y;\, m_\lambda)}$$

$$\overset{x^0<y^0}{=} \langle\Omega|\phi(y)\phi(x)|\Omega\rangle$$

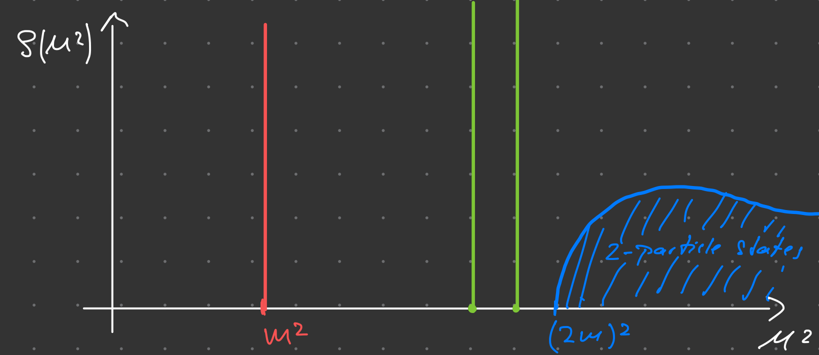
7. → Källén-Lehmann spectral representation of the two-point correlator:

$$\boxed{\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{d\mu^2}{2\pi} \rho(\mu^2)\, D_F(x-y;\mu^2)}$$

with spectral density

$$\boxed{\rho(\mu^2) = 2\pi \sum_\lambda{}' \delta(\mu^2 - m_\lambda^2) |\langle\Omega|\phi(0)|\lambda_0\rangle|^2}$$

8. Typical spectral density:



$$\rightarrow \rho(M^2) = 2\pi\, \delta(M^2 - m^2) \cdot Z + \{ \text{multi-particle states for } M^2 \gtrsim (2m)^2 \}$$

with

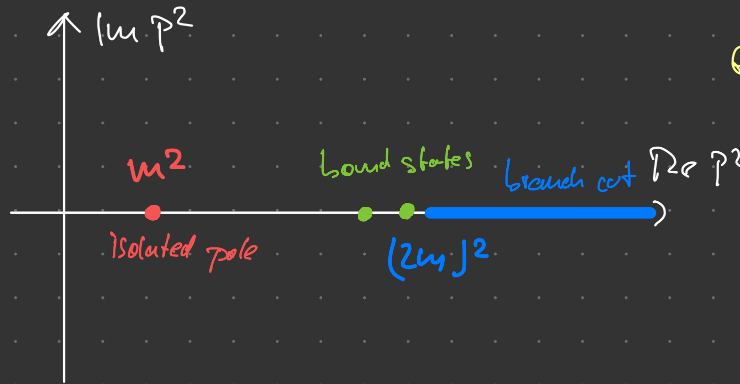
$$\boxed{\begin{array}{ll} \text{Field-strength renormalization} & Z = |\langle \Omega | \phi(0) | \lambda_0 = 1_0 \rangle|^2 \\ \text{Physical mass} & m = m_1 \\ & (\text{given by } H|1_0\rangle = m_1 |1_0\rangle) \\ \text{Bare mass} & m_0 \\ & (\text{given by } H = \ldots \tfrac{1}{2} m_0^2 \phi^2 \ldots) \end{array}} \quad (6.23)$$

- Free theory: $Z = |\langle 0|\phi(0)|\vec{p}=0\rangle_0|^2 = 1$ and $m = m_0$
- Interacting theory: $Z \neq 1$ and $m \neq m_0$
- Only m is observable
- Field-strength renormalization = Probability $|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$ that $\phi(0)$ creates the interacting single particle state $|\lambda_0\rangle$ from the interacting vacuum $|\Omega\rangle$.

9. Fourier transform the two-point correlator:

$$\int d^4x\, e^{ipx} \langle\Omega|T\phi(x)\phi(0)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi} \frac{i\rho(M^2)}{p^2-M^2+i\varepsilon} \overset{(6.23)}{=} \frac{i\cdot Z}{p^2-m^2+i\varepsilon} + \int_{\sim(2m)^2}^\infty dM^2/2\pi \frac{i\rho(M^2)}{p^2-M^2-i\varepsilon}$$

for free theory $= \dfrac{i\cdot 1}{p^2-m_0^2+i\varepsilon}$



6.4.2. Application to QED: The Electron Self-Energy

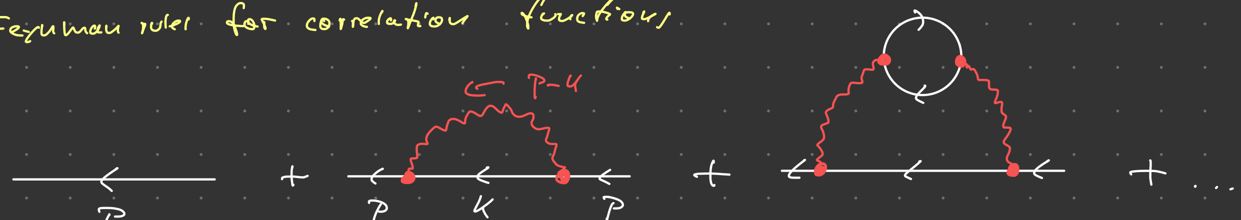
⇒ P-Set 11 for details

1. ϕ^4-theory $\mapsto$ QED: $$\int d^4x\, e^{ipx} \langle \Omega | T\, \psi(x) \overline{\psi}(0) | \Omega \rangle \overset{\circ}{=} \frac{i Z_2 (\not{p} + m)}{p^2 - m^2 + i\varepsilon} + \ldots$$

2. On the other side

$$\int d^4x\, e^{ipx} \langle \Omega | T\, \psi(x) \psi(0) | \Omega \rangle$$

Feynman rules for correlation functions

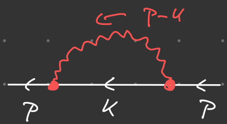
$$=$$ 

(a) (b)

3. α^0-order:

$$(a) = \frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\varepsilon}$$

4. α^1- order:



$$= \frac{i(\not p + m_0)}{p^2-m_0^2+i\varepsilon}\underbrace{\left[(-ie)^2\int\frac{d^4k}{(2\pi)^4}\,\gamma^\mu\, \frac{i(\not k+m_0)}{k^2-m_0^2+i\varepsilon}\,\gamma_\nu\, \frac{-i}{(p-k)^2+i\varepsilon}\right]}_{\equiv\, -i\Sigma_2(p)}\frac{i(\not p+m_0)}{p^2-m_0^2+i\varepsilon} \qquad (6.24)$$

→ IR- and UV-divergence

(P-Set: Photon mass μ, Pauli-Villars regularization Λ)

→ $$\Sigma_2(p) \overset{\Lambda\to\infty}{\sim} \frac{\alpha}{2\pi}\int dx\,(2m_0 - x\not p)\,\log\left[\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right] \qquad (6.25)$$

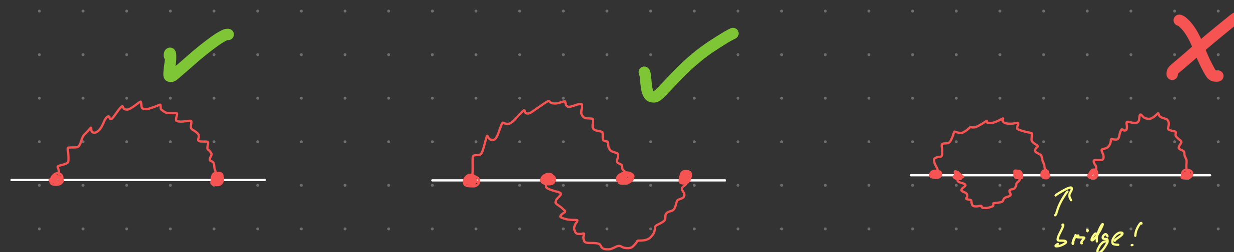
↗ branch cut emanating $p^2 = (m_0+\mu)^2$

5. Summation to all orders in α: (needed to recover isolated pole at $p^2 = m^2$)

a) Definitions:

One-particle irreducible (1PI) diagram = Bridgeless one-particle diagram

Examples:



Let furthermore

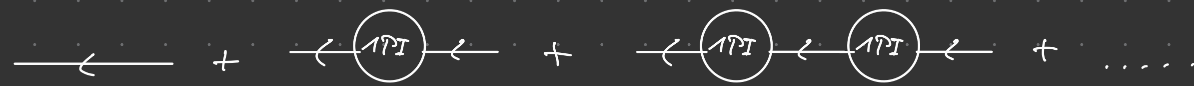
$$-i\Sigma(P) \equiv \{\text{Sum of all 1PI diagrams}\} =$$

$$= -i\Sigma_2(P) + O(\alpha^2)$$

b) Then

$$\int d^4x \, e^{ipx} \langle \Omega | T\phi(x)\phi(0) | \Omega \rangle$$

$$= \{ \text{Sum of all one-particle diagrams} \}$$

$=$ 

$$= \frac{i\cancel{(\not p + m_0)}}{p^2 - m_0^2} + \frac{i(\not p + m_0)}{p^2 - m_0^2}\left[-i\Sigma(p)\right]\frac{i(\not p + m_0)}{p^2 - m_0^2} + \dots$$

$(\not p - m_0)\cancel{(\not p + m_0)}$ and $p^2 = \not p^2$ and $\Sigma(p) = \Sigma(\not p)$ and $[\Sigma(p), \not p] = 0$

$$= \frac{i}{\not p - m_0} \sum_{n=0}^{\infty} \left(\frac{\Sigma(\not p)}{\not p - m_0} \right)^n$$

Geometric series

$$= \frac{i}{\not p - m_0} \cdot \frac{1}{1 - \frac{\Sigma(\not p)}{\not p - m_0}} = \frac{i}{\not p - m_0 - \Sigma(\not p)}$$

6. Laurent series:

$$\frac{i}{\not{p} - m_0 - \Sigma(\not{p})} \overset{!}{=} \frac{i Z_2}{\not{p} - m} + \dots$$

→ Expect simple pole for $\not{p} = m \cdot \mathbb{1} = m$

$$\boxed{m - m_0 = \Sigma(\not{p} = m)}$$

Implicit equation for physical mass m.

→ Expand denominator around this root:

$$\not{p} - m_0 - \Sigma(\not{p}) = (\not{p} - m) \cdot \left(1 - \frac{d\Sigma}{d\not{p}}\bigg|_{\not{p}=m}\right) + O\left((\not{p}-m)^2\right)$$

→ $$\boxed{Z_2 = \left(1 - \frac{d\Sigma}{d\not{p}}\bigg|_{\not{p}=m}\right)^{-1}}$$

7. Results in leading order $\mathcal{O}(\alpha)$:

a) Physical mass:

$$\boxed{\begin{aligned}\delta m = m - m_0 &= \Sigma(\not p = m)\\ &= \Sigma_2(\not p = m) + \mathcal{O}(\alpha^2)\\ &= \Sigma_2(\not p = m_0) + \mathcal{O}(\alpha^2)\\ &\quad \text{Use (6.25)} \Leftrightarrow \text{P-Set 11}\\ &\overset{\Lambda \to \infty}{\sim} \frac{3\alpha}{4\pi} \cdot m_0 \cdot \log\left(\frac{\Lambda^2}{m_0^2}\right) \xrightarrow{\Lambda \to \infty} \infty\end{aligned}}$$

→ Mass shift is UV-divergent!

b) Field strength renormalization:

$$\delta Z_2 = Z_2 - 1 = \frac{d\Sigma}{d\not p}\Big|_{\not p = m} + O(\alpha^2) = \frac{d\Sigma_2}{d\not p}\Big|_{\not p = m} + O(\alpha^2)$$

$\frac{1}{1-x} = 1 + x + O(x^2)$

$\Lambda \to \infty$

$$\overset{\text{①}}{=} \frac{\alpha}{2\pi} \int_0^1 dx \left\{ -x \log\left[\frac{x\Lambda^2}{(1-x)^2 m^2 + x\mu^2}\right] + 2(2-x)\frac{x(1-x)m^2}{(1-x)^2 m^2 + x\mu^2} \right\}$$

→ Field-strength renormalization is also UV-divergent!

Note 6.3

- $m \to \infty$ expected classically since energy of electrostatic field of sphere with radius r_e diverges as $\frac{1}{r_e} \sim \Lambda \to \infty$
- Conceptually, $m \to \infty$ is still a problem because we need to replace m_0 by m in QED predictions and $\delta m = \infty$! (⇒ renormalized perturbation theory)
- $\overset{\text{①}}{\to}$ $\delta Z_2 = -F_1^{(1)}(0)$

  where $F_1^{(1)}(0)$ is the term we subtracted from the form factor of the vertex correction.

(to ensure $F_1(0)=1$, recall (6.10)).

An application of the LSZ reduction formula yields a correction to the form factors

$$F_1(q^2) = 1 + F_1^{(1)}(q^2) + \underbrace{\delta Z_2}_{-F_1^{(1)}(0)} = 1 + F_1^{(1)}(q^2) - F_1^{(1)}(0)$$

(this justifies our ad-hoc substraction of $F_1^{(1)}(0)$)