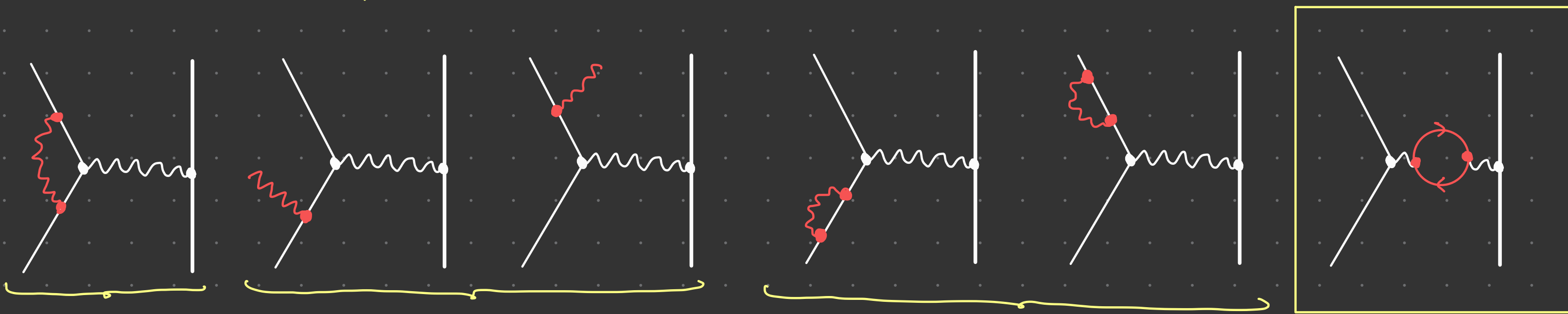
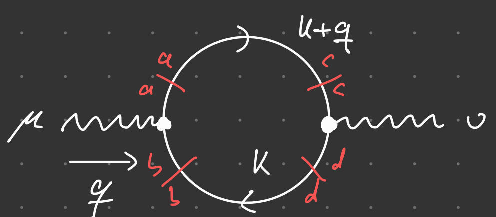
6.5 Electric Charge Renormalization



Vertex correction — Soft bremsstrahlung — Electron self-energy — Vacuum polarization $\Rightarrow$ today

Here: Vacuum polarization diagram $\rightarrow$ Photon self-energy

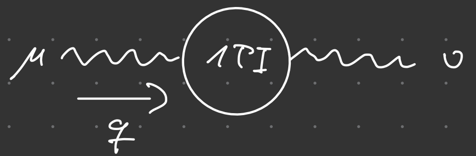
1. One-loop correction:



$$= (-1)\,(-ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left[\gamma^\mu_{ab}\, \frac{i(\not k + m)_{bd}}{k^2 - m^2}\, \gamma^\nu_{dc}\, \frac{i(\not k + \not q + m)_{ca}}{(k+q)^2 - m^2}\right]$$

$$= i\Pi_2^{\mu\nu}(q) \qquad (6.29)$$

2. Sum of all 1-particle irreducible diagrams:



$$\equiv i\Pi^{\mu\nu}(q) = i\left[\Pi_2^{\mu\nu}(q) + \mathcal{O}(\alpha^2)\right]$$

a) Only tensors available: $g^{\mu\nu}$ and $q^\mu q^\nu$ $\rightarrow$ $\Pi^{\mu\nu}(q) = A(q^2)\, g^{\mu\nu} + B(q^2)\, q^\mu q^\nu$

b) Ward identity (see (6.5)) $\xrightarrow{*}$ $q_\mu \Pi^{\mu\nu}(q) \overset{*}{=} 0$ $\rightarrow$ $B = -A/q^2$

$$\rightarrow \quad \Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \cdot \frac{A}{q^2}$$

c) $\xrightarrow{*}$ $\Pi^{\mu\nu}(q)$ has no pole for $q^2 = 0$

[Pole at $q^2 = 0$ comes from $\frac{-ig_{\mu\nu}}{q^2 + i\varepsilon}$ which is not included in 1PI diagrams]

$\rightarrow$ $\Pi(q^2) \equiv A(q^2)/q^2$ regular at $q^2 = 0$

$$\rightarrow \quad \boxed{\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \cdot \Pi(q^2)} \qquad (6.30)$$

3. Sum of all diagrams:

$$= -\frac{i g_{\mu\nu}}{q^2} + \frac{-i g_{\mu\rho}}{q^2}\, \overbrace{i\left(q^2 g^{\rho\sigma} - q^\rho q^\sigma\right)\Pi(q^2)}^{\Pi^{\rho\sigma}(q)}\, \frac{-i g_{\sigma\nu}}{q^2} + \ldots\ldots$$

Define $\Delta^\rho_\nu \equiv \delta^\rho_\nu - q^\rho q_\nu / q^2$ and use $g^{\rho\sigma} g_{\sigma\nu} = \delta^\rho_\nu$

$$= -\frac{i g_{\mu\nu}}{q^2} + \frac{-i g_{\mu\rho}}{q^2} \Delta^\rho_\nu\, \Pi(q^2) + -\frac{i g_{\mu\rho}}{q^2} \Delta^\rho_\sigma \Delta^\sigma_\nu\, \Pi^2(q^2) + \ldots$$

Use $\Delta^\rho_\sigma \Delta^\sigma_\nu = \Delta^\rho_\nu$

$$= -\frac{i g_{\mu\nu}}{q^2} + \frac{-i g_{\mu\nu}}{q^2} \Delta^\rho_\nu \underbrace{\sum_{n=1}^{\infty} \Pi^n(q^2)}$$

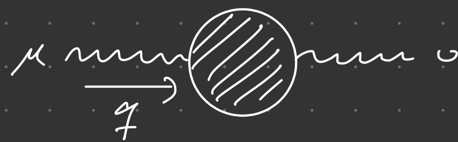
Geometric series: $\frac{1}{1-\Pi(q^2)} - 1$

$$= \frac{-i}{q^2\left[1-\Pi(q^2)\right]}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right)$$

4. If $\Pi^{\mu\nu}(q)$ contracted with a vertex to form an S-matrix element:

Ward identity $\xrightarrow{*}$

$$\boxed{\quad}$$



$$\hat{=} \quad \frac{-ig_{\mu\nu}}{q^2\left[1-\Pi(q^2)\right]}$$

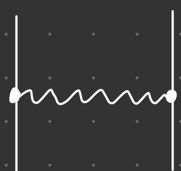
$\to$ Fixed pole at $q^2 = 0$ (to all orders of α) $\longrightarrow$ Photon remains massless

5. Charge renormalization:

a) Define

$$\boxed{Z_3 \equiv \frac{1}{1-\Pi(0)}}$$

Then, for $q^2 \to 0$:



$$= \dots \frac{e^2 g_{\mu\nu}}{q^2} \dots \longrightarrow$$

$$= \dots \frac{Z_3 e^2 g_{\mu\nu}}{q^2} \dots$$

b) $\rightarrow$ Charge renormalization:

Bare charge e_0 (given by $\mathcal{L}_{int} = e_0 \bar{\psi} \gamma^\mu \psi A_\mu$)

Physical charge $e \equiv \sqrt{Z_3}\, e_0$

Fine-structure constant $\frac{e^2}{4\pi} = \alpha \equiv Z_3 \alpha_0 = Z_3 \frac{e_0^2}{4\pi}$

Note:
- In lowest order α^0, it is $Z_3 = 1$ so that $e = e_0$, $\alpha = \alpha_0$
- In general $Z_3 = 1 + \mathcal{O}(\alpha_0) \rightarrow \alpha = \alpha_0 + \mathcal{O}(\alpha_0^2)$
- In particular $\mathcal{O}(\alpha^2) = \mathcal{O}(\alpha_0^2)$

c) In addition, for $q^2 \neq 0$ and $\Pi(q^2) = \Pi_2(q^2) + \mathcal{O}(\alpha^2)$, each virtual photon line comes with:

$$-\frac{ig_{\mu\nu}}{q^2} \cdot \frac{e_0^2}{1-\Pi(q^2)} = \frac{-ig_{\mu\nu}}{q^2} \cdot \frac{e^2[1-\Pi(0)]}{1-\Pi(q^2)} = \frac{-ig_{\mu\nu}}{q^2} \frac{e^2[1-\Pi_2(0)]}{1-\Pi_2(q^2)} + \mathcal{O}(\alpha^2)$$

Use $(1-x) = (1+x)^{-1} + \mathcal{O}(x^2)$

$$= \frac{-ig_{\mu\nu}}{q^2} \frac{e^2}{[1-\Pi_2(q^2)] \cdot [1+\Pi_2(0)]} + \mathcal{O}(\alpha^2)$$

$$= -\frac{ig_{\mu\nu}}{q^2} \frac{e^2}{1-(\Pi_2(q^2)-\Pi_2(0))} + \mathcal{O}(\alpha^2)$$

$\rightarrow$ q^2-dependent charge / fine-structure constant:

$$\boxed{\alpha_{eff}(q^2) \equiv \frac{e_0^2/4\pi}{1-\Pi(q^2)} = \frac{\alpha}{1-[\Pi_2(q^2)-\Pi_2(0)]} + \mathcal{O}(\alpha^2)}$$

6. Computation of Π_2:

Note: Replace m_0 and e_0 by m and e since $\frac{i\alpha}{\not k - m} = \frac{i\alpha_0}{\not k - m_0} + O(\alpha_0^2)$

$$i\Pi_2^{\mu\nu}(q) = (-1)\,(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{i(\not k + m)}{k^2 - m^2}\,\gamma^\nu\,\frac{i(\not k + \not q + m)}{(k+q)^2 - m^2}\right]$$

Trace identities

$$= -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu (k+q)^\nu + k^\nu (k+q)^\mu - g^{\mu\nu}(k\cdot(k+q) - m^2)}{(k^2 - m^2)\left[(k+q)^2 - m^2\right]}$$

Feynman parameter, $l \equiv k + xq$, Wick rotation $l^0 \equiv i l_E^0$

$$\overset{\circ}{=} -4ie^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{\underbrace{-\frac{2}{d} g^{\mu\nu} l_E^2 + g^{\mu\nu} l_E^2}_{\circ} - 2x(1-x) q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)}{(l_E^2 + \Delta)^2} \qquad (6.31)$$

$\circ$ (6.8) ff.
(here: $d=4$)

where $\Delta \equiv m^2 - x(1-x)q^2$

b) Strong UV-divergence: $\not\exists$ UV-cutoff $|l_E| < \Lambda$, then

$$i\Pi_2^{\mu\nu}(q) \sim e^2 \Lambda^2 g^{\mu\nu} \xrightarrow{\Lambda \to \infty} \infty$$

$\rightarrow$ To make sense of this result, a <u>regularization</u> is needed!

c) Dimensional regularization: $\Longrightarrow$ Proof P-Set 11

i) Lower the spacetime dimension $d \in \mathbb{N}$ until the UV-divergence vanishes

ii) Generalize all expressions to $d \in \mathbb{R}$

iii) Take the limit $d \nearrow 4$ in observable quantities

For $d \in \mathbb{C}$ we find:

$$\boxed{\begin{aligned} \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^n} &= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \qquad (6.32) \\ \int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta)^n} &= \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \qquad 6.33 \end{aligned}}$$

χ $n=2$: $\Gamma(z)$ has poles $z=0,-1,-2,\dots \longrightarrow \Gamma(2-\frac{d}{2})$ has isolated poles at $d=4,6,\dots$

χ $d=4-\varepsilon$ and use

Euler–Mascheroni constant

$$\Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \qquad (6.34)$$

Note: $g_{\mu\nu} g^{\mu\nu} = d$, so that in invariant integrals over spacetime the substitution

$$\boxed{l^\mu l^\nu \hat{=} \frac{1}{d} l^2 g^{\mu\nu}} \qquad (6.35)$$

is valid (generalization of (6.8))

d) Evaluate (6.31) with (6.32) & (6.33) & (6.35) (and use $z\,\Gamma(z) = \Gamma(1+z)$) $\longrightarrow$

$$\boxed{\begin{aligned} i\Pi_2^{\mu\nu}(q) &\hat{=} (q^2 g^{\mu\nu} - q^\mu q^\nu) \cdot i\Pi_2(q^2) \\ \text{with} \quad \Pi_2(q^2) &= \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{x(1-x)\,\Gamma(2-\frac{d}{2})}{\underbrace{[m^2 - x(1-x)q^2]}_{\Delta}{}^{2-d/2}} \end{aligned}}$$

e) Use (6.34) to expand in ε.

$$\Pi_2(q^2) \stackrel{\circ}{=} -\frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\left[\frac{2}{\varepsilon} - \log(\Delta) - \gamma + \log(4\pi)\right] + O(\varepsilon) \qquad (6.36)$$

7. <u>$O(\alpha)$ charge renormalization:</u>

$$\boxed{\frac{e^2 - e_0^2}{e_0^2} = Z_3 - 1 = \frac{\Pi(0)}{1-\Pi(0)} = \Pi_2(0) + O(\alpha^2) \overset{\varepsilon\to 0}{\sim} -\frac{2\alpha}{3\pi\cdot\varepsilon} \xrightarrow{\varepsilon\searrow 0} -\infty}$$

$\rightarrow$ If the observed charge is finite, $-\infty < e < 0$, the bare charge diverges, $e_0 = -\infty$.

8. $O(\alpha)$ q^2-dependence of $\alpha_{eff}(q^2)$ depends on

$$\hat{\Pi}_2(q^2) \equiv \Pi_2(q^2) - \Pi_2(0) = -\frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\log\left(\frac{m^2}{m^2 - x(1-x)q^2}\right)$$

$\rightarrow$ UV-divergence for $\varepsilon \to 0$ drops out!

9. Analysis & Interpretation of $\hat{\Pi}_2(q^2)$:

a) Pole-structure of $\hat{\Pi}_2(q^2)$ $\Rightarrow$ Script p. 131

b) Effective potential in non-relativistic limit (recall (4.18) ff.)

$$V(\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\vec{x}} \frac{-e^2}{|\vec{q}|^2\left[1-\hat{\Pi}_2(-|\vec{q}|^2)\right]} = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\vec{x}} \left(\frac{-e^2}{|\vec{q}|^2}\right)\left[1+\hat{\Pi}_2(-|\vec{q}|^2) + O(\alpha^2)\right]$$

non-relativistic $q^2 \approx -|\vec{p}-\vec{p}'|^2$
$(|\vec{q}|^2 \ll m^2)$

$$\overset{q^2 \ll m^2}{=} \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\vec{x}} \left(\frac{-e^2}{|\vec{q}|^2}\right)\cdot\left[1+\frac{\alpha}{15\pi m^2}|\vec{q}|^2\right] + O(\alpha^3)$$

$$\overset{\circ}{\approx} -\frac{\alpha}{|\vec{x}|} - \frac{4\alpha^2}{15 m^2}\,\delta^{(3)}(\vec{x})$$

$\rightarrow$ Electromagnetic force becomes much stronger at small distances

$\rightarrow$ The Coulomb potential is only a low-energy / large-distance approximation

c) Experimental verification:

Energy shift of s-orbitals in the hydrogen atom:

$$\Delta E \approx \int d^3x \, |\psi(x)|^2 \left(\frac{-4\alpha^2}{15m^2}\right) \delta^{(3)}(\vec{x}) = -\frac{4\alpha^2}{15m^2} |\psi(0)|^2 \overset{l=0}{<} 0 \qquad (*)$$

Note: • Darwin term $\quad H_{\text{Darwin}} = \frac{\pi\alpha}{2m^2} \delta^{(3)}(\vec{x})$ (linear in α, but (*) quadratic in α!)

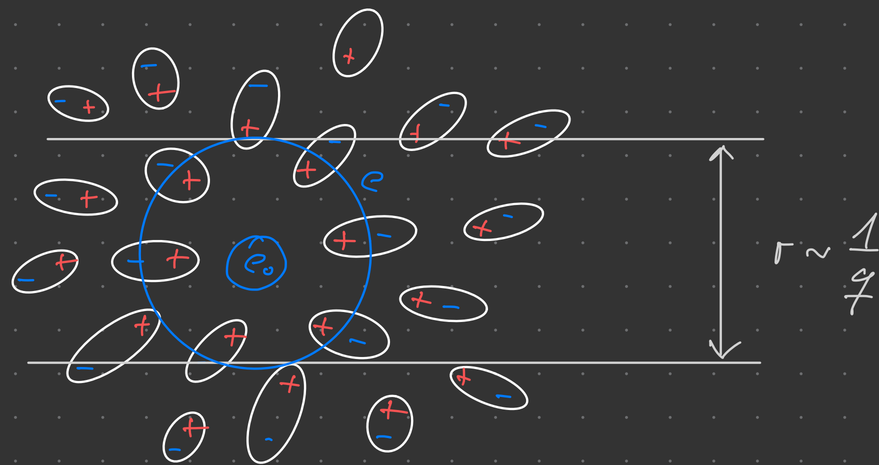
- Dirac theory (⇒ P-Set 4) → $2S_{1/2}$ and $2P_{1/2}$ are degenerate
- Experimentally splitting of $2S_{1/2}$ and $2P_{1/2}$ observed → Lamb shift
- (*) contributes 2% to the Lamb shift (pure QED effect!)

d) More generally, one finds the Uehling potential:

$$V(r) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2mr}}{(mr)^{3/2}} + \ldots\right)$$

→ Compton wavelength: $\lambda_c = \frac{h}{mc} = \frac{2\pi}{m}$, Bohr radius: $a_0 = \frac{\lambda_c}{2\pi\alpha} \approx 22 \cdot \lambda_c$ → $\delta^{(3)}$ good approximation for atomic physics

e) Interpretation: Vacuum polarization:



→ Interacting vacuum behaves as dielectric medium and screens the bare charge

→ For $r \lesssim \frac{1}{m}$ the screening becomes weaker and $e \to e_0$

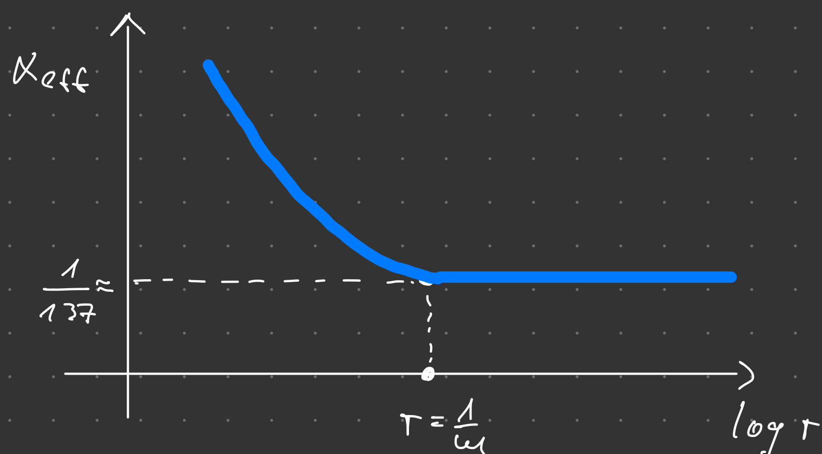
f) Relativistic limit $-q^2 \gg m^2$:

$$\hat{\Pi}_2(q^2) \overset{\circ}{=} \frac{\alpha}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right]$$

→ "Running" of α_{eff} with length scale $r = \frac{1}{q} \to 0$

$$\boxed{\alpha_{eff}(q^2) \overset{-q^2 \gg m^2}{\approx} \frac{\alpha}{1 - \frac{\alpha}{3\pi} \log\left(\frac{-q^2}{A m^2}\right)}}$$

with $A = \log\left(\frac{5}{3}\right)$



→ Renormalization (→ next two lectures)