Example 1.4: Scale transformations

1. $x' = x$ and $\phi'(x') = x^d \phi(x) = x^{-d} \phi(x)$

2. $T(\phi) = x^{-d} \phi$, $\frac{\partial x'^0}{\partial x^\mu} = x^{-1} \delta^\mu_0 \Rightarrow \left| \frac{\partial x'}{\partial x} \right| = x^d$

3. Action: $S[\phi] = \int d^d x \, \mathcal{L}(x^{-d} \phi(x), x^{-1-d} \partial_\mu \phi(x))$

   $\mathcal{L}[\phi] = \frac{1}{2} \int d^d x \, (\partial \phi)^2$

   $= \lambda d^{-2-2\Delta} \int d^d x \, \mathcal{L}(\phi(x), \partial \phi(x)) = \lambda d^{-2-2\Delta} S[\phi]$

   $\Rightarrow S' = S \text{ iff } \Delta = \frac{d}{2} - 1 \quad (\rightarrow \text{conformal field theory})$

Example 1.5: Phase transformations

1. $x' = x$ and $\phi'(x') = e^{i\Theta} \phi(x)$

2. $T(\phi) = e^{i\Theta} \phi$ and $\frac{\partial x'^0}{\partial x^\mu} = \delta^\mu_0 \Rightarrow \left| \frac{\partial x'}{\partial x} \right| = 1$
Infinitesimal Transformations

1. Infinitesimal transformations:

\[ x'^\mu = x^\mu + \omega^\mu \cdot \frac{\delta x^\mu}{\delta \omega^\nu} (x) \quad \text{and} \quad \phi'(x') = \phi(x) + \omega^\nu \cdot \frac{\delta \phi}{\delta \omega^\nu} (x) \]  

(1.2)

2. Generator of $\Gamma$:

\[ \delta \phi(x) \equiv \phi'(x) - \phi(x) \equiv -i \omega^\nu \cdot \partial_\nu \phi(x) \]

With

\[ \phi'(x') = \phi(x) + \omega^\nu \cdot \frac{\delta \phi}{\delta \omega^\nu} (x) = \phi(x) - \omega^\nu \cdot \frac{\delta x^\mu}{\delta \omega^\nu} \partial_\mu \phi(x') + \omega^\nu \frac{\delta \phi}{\delta \omega^\nu} (x') + O(\omega^2) \]

it follows

\[ i \partial_\nu \phi = \frac{\delta x^\mu}{\delta \omega^\nu} \partial_\mu \phi - \frac{\delta \phi}{\delta \omega^\nu} \]

Example 1.6: Translations

1. \[ x'^\mu = x^\mu + \omega^\mu = x^\mu + \omega^\mu \cdot \frac{\delta x^\mu}{\delta \omega^\nu} \]

\[ \Rightarrow \quad i \partial_\nu \phi = 5 \mu \partial_\nu \phi - 0 \quad \text{and therefore} \]

\[ \partial_\mu = -i \partial_\mu = \gamma_\mu \]
Example 1.7: Scale Transformations

\[ G_\mu = -i x^\mu \partial_\mu \equiv D \quad \text{"dilations"} \]

\[ \mu = 1, 2, 3 \]

Example 1.8: Spatial Relations

\[ G_{\mu \nu} = i (\partial_\mu x - x_\mu \partial_\nu) + \delta_{\mu \nu} \]

\[ G_{12} = i (x_1 \partial_2 - x_2 \partial_1) \quad \delta_{1 \nu}, \delta_{1 \nu}, \delta_{2 \nu} \]

\[ \sum_{\nu = 1}^{3} = 2 \]

---

Noether's Theorem

1. Transformation (1.2) is symmetry of the action \( \iff S[\phi] = S[\phi'] \) (for \( \omega \) independent)

2. Assume that (1.2) not rigid: \( \omega_\alpha = \omega_\alpha(x) \)

3. Jacobian:

\[ \frac{\partial x^{10}}{\partial x^\mu} = \delta^0_\mu + \delta_\mu (\omega_\alpha \frac{\partial x^0}{\partial \omega_\alpha}) \quad \Rightarrow \quad \left| \frac{\partial x^i}{\partial x^j} \right| = 1 + \delta_\mu (\omega_\alpha \frac{\partial x^0}{\partial \omega_\alpha}) \]

\[ \text{det}(A + A) = 1 + Tr[A^2] + O(A^2) \]
4. Inverse Jacobian: \( \frac{\partial x^0}{\partial x^1} = \frac{\partial x^0}{\partial \omega_0} = \frac{\partial x^0}{\partial \omega_0} \frac{\partial \omega_0}{\partial x^1} \)

5. Use (1.1):
\[
S' = \int d^d x \left[ 1 + \phi \left( \frac{\partial x^0}{\partial \omega_0} \right) \right] \right( \phi + \phi_0 \frac{\partial \omega_0}{\partial \omega_0} \right) \right( \frac{\partial x^0}{\partial \omega_0} \right) \times \left[ \frac{\partial \phi}{\partial \omega_0} + \phi_0 \left( \frac{\partial x^0}{\partial \omega_0} \right) \right] \]

6. Expand in 1st order of \( \omega_0 \) and \( \omega_1 \):
\[
\frac{\partial \omega_0}{\partial x^1} \leq \left( \frac{\partial \omega_0}{\partial x^1} \right)
\]

7. \( \delta S = S' - S \rightarrow \) Only terms as \( \frac{\partial \omega_0}{\partial x^1} \) remain

8. For generic, non-rigid transformation, we find
\[
\delta S = - \int d^d x \ j^x \ dw
\]

with the current
\[
j^x = \left\{ \frac{\partial x^0}{\partial (\partial_0 \phi)} \right\} \frac{\partial x^0}{\partial \omega_0} - \frac{\partial x^0}{\partial (\partial_0 \phi)} \cdot \frac{\partial x^0}{\partial \omega_0} \tag{1.3}
\]

9. Integration by parts: \( \delta S = \int d^d x \ \omega_0 \ dw \ j^x \)
10. Let \( \phi \) obey the equations of motion \( \delta S = 0 \) for arbitrary variations \( \phi' = \phi + \delta \phi \).

In particular, for arbitrary non-rigid transformations \( \omega_0(x) \! : \! \nu \to \nu' \):

It follows **Noether's (first) theorem**:

\[
\partial_{\nu} j^{\mu}_a = 0 \quad \forall a, \chi
\]

\[
\delta_t j^0 - \nabla j^0 = 0
\]

\[
\delta_t j^0 = \nabla j^0
\]

11. **Conserved charge**

\[
Q_a = \int_{\text{space}} d^{d-1} x \ j^0_a
\]

Indeed

\[
\frac{dQ_a}{dt} = \int_{\text{space}} d^{d-1} x \ \partial_\nu j^0_a = - \int_{\text{space}} d^{d-1} x \ \partial_\nu j^\mu_a \quad \text{Noether}
\]

\[
\int_{\text{surface}} d\Sigma \ n^\nu j^\mu_a = 0 \quad \text{Gauss}
\]
Note 1.1

The current (1.3) is called canonical current as there is an ambiguity,
\[ j'_\mu = j_\mu + \partial_\nu B_\nu^\mu \text{ with } B_\nu^\mu = -B_\nu^\mu \text{ arbitrary } \Rightarrow \partial_\mu j'_\mu = 0 \]

Note 1.2

Symmetric Lagrangian $\Rightarrow$ Symmetric action $\Rightarrow$ Symmetric EOMs

$\Rightarrow$ Conserved currents

Application: The Energy-Momentum Tensor (EMT)

1. $\xi$ infinitesimal: $x'\mu = x\mu + \varepsilon\xi^\mu \Rightarrow \frac{\delta x'\mu}{\delta \varepsilon} = \delta x_\mu, \frac{\delta S}{\delta \varepsilon} = 0$

2. $\xi$ translation invariant action: $S' = S$
3. **Conserved currents:**

\[
T^\mu_\nu = \left\{ \frac{\partial x}{\partial \phi} \right\} \partial \phi - \delta^\mu_\nu x \]

\[
T^\mu_\nu = \frac{\partial x}{\partial \phi} \theta \phi - g^\mu_\nu x
\]

*Energy-Momentum Tensor*

\[
T^0_0 = \int d^3x \ T^0_0
\]

4. **Energy** \((\omega = 0)\)

\[
T^0_0 = \int d^3x \ T^0_0 = \int d^3x \left\{ \frac{\partial x}{\partial \phi} \phi - x \right\}^2 = \int d^3x \ \partial (\phi x) = H
\]

with \(\partial \mu T^{\mu 0} = 0\) and four conserved charges

\[
T^0_0 = \int d^3x \ T^0_0
\]
5. Kinetic momentum ($\omega = i$):

$$P^i = \int d^3x \phi^0 i = \int d^3x \frac{\partial \phi}{\partial \phi} (-\partial_i \phi) = -\int d^3x \pi^i \partial_i \phi$$

**Note 1.3**

In general $T^{\mu \nu} \neq T_{\mu \nu}$ for the canonical EMT. But:

$$\tilde{T}^{\mu \nu} = T^{\mu \nu} + \partial \eta^{\mu \nu}$$

Choose $\eta^{\mu \nu}$ such that $\tilde{T}^{\mu \nu} = \hat{T}^{\mu \nu}$ ($\rightarrow$ Relativistic EMT)

**Example 1.9: Electromagnetism in the vacuum**

1. **Four-component field**: $A^\mu = (\phi, A^1, A^2, A^3)$

2. **EM field tensor**: $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

3. **Lagrangian**: $L_{\text{em}} (A^\mu, \partial_\mu A^\nu) = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu}$
4. **Action:** \( \mathcal{S}_m = \int dt x \mathcal{L}_m \)

5. **Euler-Lagrange equations:**
   \[ \partial_\mu \mathcal{F}^{\mu\nu} = 0 \]
   (inhomogeneous Maxwell equation)

6. \( \mathcal{S}_m \) is Lorentz invariant and translation invariant → \( \text{EMT} = \text{conserved currents} \)

7. **Canonical EMT:**
   \[ T^{\mu\nu}_{\text{em}} = \frac{\partial \mathcal{L}_m}{\partial (\partial_\mu A_\nu)} - g^{\mu\nu} \mathcal{L}_m \]

8. **Symmetric EMT via** \( K_{\mu\nu} = F^{\mu\lambda} A^\lambda :\)
   \[ \mathcal{F}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} \mathcal{F}_\lambda \mathcal{F}^{\lambda\rho} - \mathcal{F}_\mu \mathcal{F}^{\nu} \]

   \* \( T^{\infty \infty} = \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) \) (energy density)

   \* \( T^{\infty i} = (\mathbf{E} \times \mathbf{B})_i \) (Poynting vector)

   \* \( T^{ij} = \sigma_{ij} \) (Maxwell stress tensor)

Details \( \square \) Problem Set 1