Example 1.4: Scale transformations

scaling dimension

1. $x' := \lambda \cdot x$ and $\phi'(x') := \lambda^{-\Delta}\phi(x) = \lambda^{-\Delta}\phi(\lambda^{-1}x')$

2. $\mathcal{F}(\phi) = \lambda^{-\Delta}\phi\,, \quad \dfrac{\partial x^\nu}{\partial x'^\mu} = \lambda^{-1}\delta^\nu_\mu \Rightarrow \left|\dfrac{\partial x'}{\partial x}\right| = \lambda^d$

3. Action: $S[\phi'] \overset{(1.1)}{=} \lambda^d \int d^dx\, \mathcal{L}\left(\lambda^{-\Delta}\phi(x),\, \lambda^{-1-\Delta}\partial_\mu\phi(x)\right)$

$S[\phi] = \frac{1}{2}\int d^dx\, (\partial_\mu\phi)^2$

$$= \lambda^{d-2-2\Delta}\int d^dx\, \mathcal{L}(\phi(x), \partial_\mu\phi(x)) = \lambda^{d-2-2\Delta}\, S[\phi]$$

$\Rightarrow S' = S$ iff $\Delta = \frac{d}{2} - 1$ $\quad$ ($\rightarrow$ conformal field theory)

Example 1.5: Phase rotations

1. $x' := x$ and $\phi'(x') := e^{i\theta}\phi(x)$

2. $\mathcal{F}(\phi) = e^{i\phi}\phi$ and $\dfrac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu \Rightarrow \left|\dfrac{\partial x'}{\partial x}\right| = 1$

Infinitesimal Transformations

1. Infinitesimal transformations:

$$x'^{\mu} = x^{\mu} + \omega_a \cdot \frac{\delta x^{\mu}}{\delta \omega_a}(x) \quad \text{and} \quad \phi'(x') = \phi(x) + \overset{\omega_a(x)}{\omega_a} \cdot \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \qquad (1.2)$$

2. Generator of IT:

$$\delta_\omega \phi(x) \equiv \phi'(x) - \phi(x) \equiv -i\, \omega_a\, G_a \phi(x)$$

With

$$\phi'(x') = \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) = \phi(x') - \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \partial_\mu \phi(x') + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x') + \mathcal{O}(\omega^2)$$

it follows

$$\boxed{i G_a \phi = \frac{\delta x^{\mu}}{\delta \omega_a} \partial_\mu \phi - \frac{\delta \mathcal{F}}{\delta \omega_a}}$$

Example 1.6: Translations

$$\left.\begin{array}{l} 1.\quad x'^{\mu} = x^{\mu} + \omega^{\mu} = x^{\mu} + \omega^{\nu} \dfrac{\delta x^{\mu}}{\delta \omega^{\nu}} \\ 2.\quad \dfrac{\delta \mathcal{F}}{\delta \omega^{\nu}} = 0 \qquad \Rightarrow \quad \dfrac{\delta x^{\mu}}{\delta \omega^{\nu}} = \delta^{\mu}_{\nu} \end{array}\right\}$$

3. $i G_\mu \phi = \delta^{\nu}_{\mu} \partial_\nu \phi - 0$ and therefore

$$G_\mu = -i \partial_\mu = P_\mu$$

Example 1.7: Scale Transformations

$$G = -i x^\mu \partial_\mu \equiv D \qquad \text{"dilations"}$$

$\mu = 1,2,3$

Example 1.8: Spatial Rotations

$$G_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}$$

$$G_{12} = \underbrace{i(x_1 \partial_2 - x_2 \partial_1)}_{L_{3=z}} \qquad S_x, S_y, S_z$$

Noether's Theorem

1. Transformation (1.2) is symmetry of the action $:\Longleftrightarrow S[\phi] = S[\phi']$ (for ω_a independent for x $\Rightarrow$ rigid transformation)
2. Assume that (1.2) not rigid: $\omega_a = \omega_a(x)$
3. Jacobian: $\dfrac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\mu + \partial_\mu\left(\omega_a \dfrac{\delta x^\nu}{\delta \omega_a}\right) \longrightarrow \left|\dfrac{\partial x'}{\partial x}\right| = 1 + \partial_\mu\left(\omega_a \dfrac{\delta x^\mu}{\delta \omega_a}\right)$

$$\det(\mathbb{1} + A) = 1 + \mathrm{Tr}[A] + \mathcal{O}(A^2)$$

4. Inverse Jacobian: $\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)$

5. Use (1.1):

$$S' = \int d^d x \left[1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)\right] \mathcal{L}\left(\phi + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}, \left[\delta^\nu_\mu - \partial_\mu \omega_a \frac{\delta x^\nu}{\delta \omega_a}\right] \times \left[\partial_\nu \phi + \partial_\nu\left(\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}\right)\right]\right)$$

6. Expand in 1st order of ω_a and $\frac{\partial \omega_a}{\partial x^\mu}$ $\leftarrow \left(\frac{\partial \omega_a}{\partial x^\mu} \ll 1\right)$

7. $\measuredangle$ $\delta S \equiv S' - S$ $\longrightarrow$ Only terms $\propto$ $\frac{\partial \omega_a}{\partial x^\mu}$ remain

8. For generic, non-rigid transformation we find

$$\delta S = -\int d^d x \; j^\mu_a \, \partial_\mu \omega_a$$

with the current

$$\boxed{j^\mu_a = \left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \frac{\delta \mathcal{F}}{\delta \omega_a}} \qquad (1.3)$$

9. Integration by parts: $\delta S = \int d^d x \, \omega_a \, \partial_\mu j^\mu_a$

10. Let ϕ obey the equations of motion $\longrightarrow$ $\delta S = 0$ for arbitrary variations $\phi' = \phi + \delta\phi$

In particular, for arbitrary non-rigid transformations $\omega_a(x)$!

It follows Noether's (first) theorem:

$$\boxed{\partial_\mu j^\mu_a = 0 \qquad \forall_{a,x}}$$

$\partial_t j^0 - \nabla \vec{j} = 0$

$\partial_t j^0 = \nabla \vec{j}$

11. Conserved charge

$$\boxed{Q_a = \int_{\text{Space}} d^{d-1}x \; j^0_a}$$

Indeed

$$\frac{dQ_a}{dt} = \int_{\text{Space}} d^{d-1}x \; \partial_0 j^0_a \overset{\text{Noether}}{=} -\int_{\text{Space}} d^{d-1}x \; \partial_k j^k_a \overset{\text{Gauss}}{=} -\int_{\text{Surface}} d\sigma_k \, j^k_a = 0$$

Note 1.1

The current (1.3) is called <u>canonical current</u> as there is an ambiguity:

$$\tilde{j}^\mu_a := j^\mu_a + \partial_\nu B^{\mu\nu}_a \quad \text{with} \quad B^{\mu\nu}_a = -B^{\nu\mu}_a \quad \text{arbitrary} \Rightarrow \partial_\mu \tilde{j}^\mu_a = 0$$

Note 1.2

$$\underbrace{\text{Symmetric Lagrangian} \Rightarrow \text{Symmetric action}}_{\longrightarrow \text{ Conserved currents}} \Rightarrow \text{Symmetric EOMs}$$

Application: The Energy-Momentum Tensor (EMT)

1. $\exists$ infinitesimal: $x'^\mu = x^\mu + \varepsilon^\mu \longrightarrow \frac{\delta x^\mu}{\delta \varepsilon^\nu} = \delta^\mu_\nu \, , \; \frac{\delta F}{\delta \varepsilon^\nu} = 0$

2. $\exists$ translation invariant action: $S' = S$

3. Conserved currents:

$$T^{\mu}{}_{\nu} = \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi - \delta^{\mu}_{\rho} \mathcal{L} \right\} \overbrace{\frac{\delta x^{\rho}}{\delta \varepsilon^{\nu}}}^{\delta^{\rho}_{\nu}} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^{\mu}_{\nu} \mathcal{L}$$

$$\boxed{T^{\mu\nu} = g^{\nu\rho} T^{\mu}{}_{\rho} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}}$$

Energy-Momentum Tensor

with $\partial_\mu T^{\mu\nu} = 0$ and four conserved charges

$Q_{a \to \nu}$

$$\boxed{P^{\nu} = \int d^3x \, T^{0\nu}}$$

4. Energy ($\nu = 0$)

$$P^{0} = \int d^3x \, T^{00} = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \right\} = \int d^3x \, \mathcal{H}(\phi, \pi) = H$$

5. Kinetic momentum $(\nu = i)$:

canonical momentum

$$P^i = \int d^3x \, T^{0i} = \int d^3x \, \frac{\partial \mathcal{L}}{\partial \dot\phi}(-\partial_i \phi) = -\int d^3x \, \pi \, \partial_i \phi$$

Note 1.3

In general $T^{\mu\nu} \neq T^{\nu\mu}$ for the canonical EMT. But:

$\tilde{T}^{\mu\nu} := T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu}$ with $K^{\rho\mu\nu} = -K^{\mu\rho\nu}$

Choose $K^{\rho\mu\nu}$ such that $\tilde{T}^{\mu\nu} = \tilde{T}^{\nu\mu}$ ($\rightarrow$ Belinfante EMT)

Example 1.9: Electromagnetism in the vacuum

1. Four-component field: $A^\mu = (\phi, A^1, A^2, A^3)$

2. EM field tensor: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

3. Lagrangian: $\mathcal{L}_{em}(A^\nu, \partial_\mu A^\nu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

4. Action: $S_{em} = \int d^4x \, \mathcal{L}_{em}$

5. Euler-Lagrange equations: $\boxed{\partial_\mu F^{\mu\nu} = 0}$ (inhomogeneous Maxwell equations)

6. S_{em} is Lorentz invariant and translation invariant $\longrightarrow$ EMT = conserved currents

Poincaré invariant

7. Canonical EMT: $T^{\mu\nu}_{em} = \dfrac{\partial \mathcal{L}_{em}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L}_{em}$

8. Symmetric EMT via $K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$:

$$\boxed{\tilde{T}^{\mu\nu}_{em} = \frac{1}{4} g^{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - F^{\mu\rho} F^{\nu}{}_{\rho}}$$

- $\tilde{T}^{00} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$ (energy density)
- $\tilde{T}^{0i} = (\vec{E} \times \vec{B})_i$ (Pointing vector)
- $T^{ij} = \sigma_{ij}$ (Maxwell stress tensor)

Details $\Rightarrow$ Problem Set 1