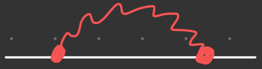
7. Systematics of Renormalization

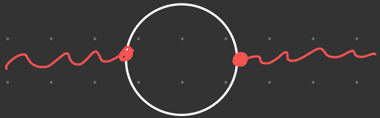
Remember:

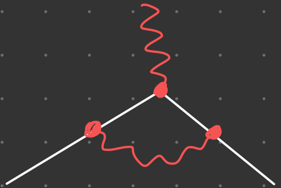
- IR-divergences:
  - Due to massless particles (photon)
  - Regulate with small photon mass μ
  - Divergences from soft virtual photons (vertex correction) and soft bremsstrahlung cancel

} Not a fundamental problem

- UV-divergences
  - Due to unbounded high momenta of particles (= unbounded small length scales) in all three radiative corrections:







  - Regulate with additional heavy particles (Λ) or dimensional regularization (ε)
  - Cancelled in some observable quantities
  - Diverging differences between physical and bare quantities

→ Fundamental problem → Study UV-divergences systematically

7.1 Counting UV-Divergences

1. Goal: Classify UV-divergences in QED

2. Definitions:

N_e = # external e^- lines

N_γ = # external γ lines

P_e = # e^- propagators $\longrightarrow \prod_{i=1}^{P_e} \frac{1}{\cancel{k}_i - m}$

P_γ = # γ propagators $\longrightarrow \prod_{i=1}^{P_\gamma} \frac{1}{k_i^2}$

V = # vertices

L = # independent loops $\longrightarrow \prod_{i=1}^{L} \int \frac{d^4 k_i}{(2\pi)^4}$

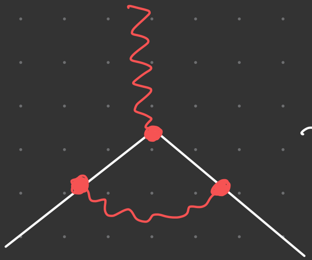
3. Superficial degree of divergence:

$$\boxed{D_{QED} = (3L + L) - (P_e + 2P_\gamma)}$$

→ **Intuition:**

$$D_{QED} \begin{cases} > 0 & : \text{Diverges with } \Lambda^D \\ = 0 & : \text{Divergence with } \log \Lambda \\ < 0 & : \text{No divergence} \end{cases}$$

Example:

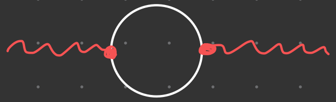


$\sim \log \Lambda$ and $D_{QED} = 4\cdot 1 - (2 + 2\cdot 1) = 0$

However: Not always correct!

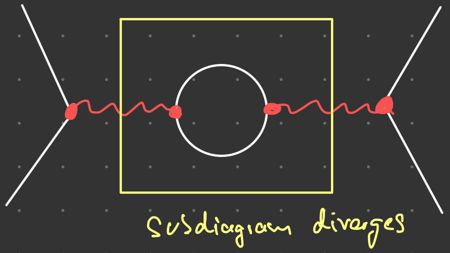
Reasons:

- Divergence may be weaker (or absent) if symmetries make divergent terms cancel:



$\sim \log \Lambda$ although $D = 4\cdot 1 - (2+2+0) = 2$

- Divergence may be worse if diagram contains divergent subdiagram:



$\sim \log \Lambda$ although $D = 4\cdot 1 - (2 + 2\cdot 2) = -$

- Tree-level diagrams without propagators have $D = 0$ but no divergence:

~ 1 although $D = 4 \cdot 0 - (0 + 2 \cdot 0) = 0$

4. Use $L = P_e + P_\gamma - V + 1$ = Cycle space dimension (7.1)

$V = 2P_\gamma + N_\gamma = \frac{1}{2}(2P_e + N_e)$ (7.2)

to show

$$\boxed{D_{QED} \overset{\circ}{=} 4 - N_\gamma - \frac{3}{2} N_e}$$

→ Independent of number of vertices!

5. <u>Aside</u>: Furry's theorem

> Feynman diagrams with an odd number of photons as their only external lines vanish identically.

Proof: Follows from charge conjugation symmetry (C) of QED.

6. Enumerate diagrams with $D_{QED} \geq 0$:

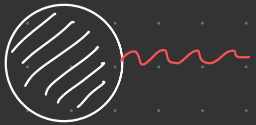
a) $N_e = 0$

i) $N_\gamma = 0$ $(D = 4)$

$\sim$ badly divergent

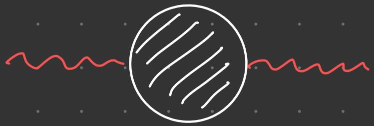
→ Unobservable vacuum energy shift → Ignore!

ii) $N_\gamma = 1$ $(D = 3)$



$\overset{\text{Furry}}{=} 0$

iii) $N_\gamma = 2$ $(D = 2)$; Recall our first-order result (6.36):

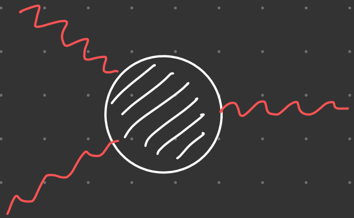


$$= (g^{\mu\nu} q^2 - q^\mu q^\nu) \cdot \Pi(q^2)$$

$$\sim (g^{\mu\nu} q^2 - q^\mu q^\nu) \cdot \frac{\text{const.}}{\varepsilon}$$

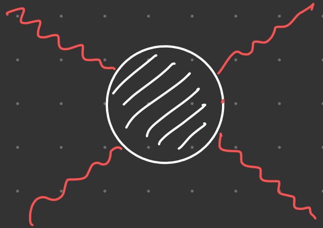
$$\sim (g^{\mu\nu} q^2 - q^\mu q^\nu) \cdot \underbrace{\text{const.} \cdot \log \Lambda}_{a_0(\Lambda)} \qquad (7.3)$$

iv.) $N_\gamma = 3 \quad (D=1)$



$$\overset{\text{Furry}}{=} \quad 0$$

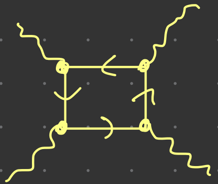
v.) $N_\gamma = 4 \quad (D=0)$



$$\overset{\text{Ward}}{\sim} \quad 1$$

Notes:

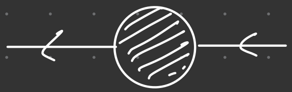
- Light-by-light scattering (Halpern scattering)
- $\sigma_{\gamma\gamma\to\gamma\gamma} \sim \left(\frac{\alpha^2}{m_e^4}\right)^2$
- Lowest order:



$$\sim \frac{\alpha^2}{m_e^4}$$

b) $N_e = 2$:

i.) $N_\gamma = 0$ $(D=1)$; Recall our first-order result in (6.28):



$$\sim \underbrace{\text{const.}\ \log\Lambda}_{a_1(\Lambda)} + \underbrace{\not{p}\cdot\text{const.}\ \log\Lambda}_{a_2(\Lambda)} \qquad (7.4)$$

ii.) $N_\gamma = 1$ $(D=0)$; Recall our first-order result in (6.12):

$$\sim \underbrace{-ie\gamma^\mu \log\Lambda}_{a_3(\Lambda)} \qquad (7.5)$$

→ Diagrams only diverge if they contain (7.3), (7.4), or (7.5) as <u>subdiagram</u>

→ QED only contains <u>four</u> UV-divergent numbers: a_0, a_1, a_2, a_3

7. <u>Idea:</u> Absorb finite number of diverging quantities in finite number of diverging but unobservable Lagrangian parameters → Renormalization (➲ next lecture)

8. <u>Generalization:</u> & QED in *d* spacetime dimensions →

$$\boxed{\begin{aligned} D_{QED} &= dL - P_e - 2P_\gamma \\ &\Big\downarrow \ (7.1)\ \&\ (7.2) \\ &\overset{\circ}{=} d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma - \left(\frac{d-1}{2}\right)N_e \end{aligned}}$$

→ <u>Observations:</u>

- $d < 4$, diagrams of higher order ($V \to \infty$) are always superficially <u>convergent</u>
- $d = 4$, D_{QED} is independent of the order of V
- $d > 4$, diagrams of higher order ($V \to \infty$) are always superficially <u>divergent</u>

9.

- Super-Renormalizable theory:

Only a finite number of Feynman diagrams (!) superficially diverge.

Example: QED in $d=2+1$

- Renormalizable theory:

Only a finite number of amplitudes (!) superficially diverge.

→ Divergences at all orders in perturbation theory

Example: QED in $d=3+1$

- Non-Renormalizable theory:

All amplitudes diverge at sufficiently high order in perturbation theory.

Example: QED in $d=4+1$

Alternative approach:

1. & ϕ^n- theory : $\mathcal{L}_{\phi^n} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{n!}\phi^n \quad n \in \mathbb{N} \quad (7.6)$

2. Definitions:

N_ϕ = # external lines

P_ϕ = # propagators

V = # Vertices

L = # independent loops

3. Superficial degree of divergence:

$$\boxed{\begin{aligned} D_{\phi^n} &\equiv dL - 2P_\phi \\ &\overset{\circ}{=} d + \left[n\left(\frac{d-2}{2}\right) - d\right] \cdot V - \left(\frac{d-2}{2}\right) N_\phi \end{aligned}} \qquad (7.7)$$

$\rightarrow$ For $n=4$ in $d=4$ independent of V $\rightarrow$ renormalizable

4. Alternative approach via dimensional analysis:

a) Recall $\hbar = c = 1$ and $\lambda_c = \frac{h}{mc} = \frac{2\pi}{m}$

$\rightarrow$ Dimension of length $[\lambda_c] = M^{-1}$ (M: dimension of mass)

b) $[S] = 1$ since $\hbar = 1$

c) $S = \int d^dx \, \mathcal{L}$ and $[d^dx] = M^{-d}$

$\rightarrow$ $[\mathcal{L}] = M^d$

d) From (7.6) and using $[\partial] = M$:

$$[\phi] = M^{\frac{d-2}{2}}$$

$$[m] = M \qquad \text{(consisten!)}$$

$$[\lambda] = M^{d-4(d-2)/2}$$

e) ∡ Amplitude $\mathcal{M}$ of single diagram with N_ϕ external lines

→ Could arise (on tree-level) from interaction $\eta\,\phi^{N_\phi}$ → $[\eta] = \mu^{d-N_\phi(d-2)/2}$

$$\to [\mathcal{M}] \underset{\uparrow}{=} [\eta] = \mu^{d-N_\phi(d-2)/2}$$

Recall: $\mathcal{M} = -\lambda + O(\lambda^2)$ from Eq. (4.16)

f) ∡ Diagram with V vertices ⟶ $\mathcal{M} \sim \lambda^V \Lambda^D$ for UV-cutoff $\Lambda \to \infty$

Use $[\Lambda] = \mu$ →

$$[\lambda]^V [\Lambda]^D = [\mathcal{M}] = \mu^{d-N_\phi(d-2)/2}$$

$$\Rightarrow \quad V \cdot \underbrace{\log_\mu [\lambda]}_{\text{"mass dimension of } [\lambda]\text{"}} + D = d - N_\phi \left(\frac{d-2}{2}\right)$$

$$\to \boxed{D_{\phi^4} = d - \underbrace{\log_\mu [\lambda]}_{d-4\frac{d-2}{2}} \cdot V - \left(\frac{d-2}{2}\right) N_\phi = (7.7)}$$

5. Equivalent characterization:

- **Super-Renormalizable theory:**

  Coupling constant has positive mass dimension: $\log_{\mu}[\lambda] > 0$

- **Renormalizable theory:**

  Coupling constant is dimensionless: $\log_{\mu}[\lambda] = 0$

  Example: QED with $[e] = 1$ is superficially renormalizable.

- **Non-Renormalizable theory:**

  Coupling constant has negative mass dimension: $\log_{\mu}[\lambda] < 0$

Aside: Why quantum gravity is special

1. Fields: Components of the metric tensor $g_{\mu\nu}(x)$

2. Einstein-Hilbert action of pure gravity:

$$\boxed{S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{|\det g(x)|}\left[R(g(x)) - 2\Lambda_c\right]} \qquad (7.8)$$

$R = g^{\mu\nu} R_{\mu\nu}$: Ricci scalar with Ricci tensor $R_{\mu\nu}$

Λ_c : Cosmological constant

G: Gravitational constant = Coupling constant of gravity

⇒ Equations of motion = Einstein's field equations (in vacuum):

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_c g_{\mu\nu} = 0}$$

3. Recall: $R \sim g^{\mu\nu} R_{\mu\nu} \sim g^{\mu\nu} R^{\sigma}{}_{\mu\sigma\nu} \sim g^{\mu\nu} \partial_\sigma \Gamma^{\sigma}{}_{\mu\nu} \sim g^{\mu\nu} \partial_\sigma \left(g^{\sigma\kappa} \partial_\mu g_{\nu\kappa} \right)$

Riemann curvature tensor

Christoffel symbol (of 2nd kind)

$$\Rightarrow [R] = [g]^3 [\partial]^2$$

and $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$$\Rightarrow \cancel{L^2} = [ds^2] = [g]\,[dx]^2 = [g]\,\cancel{L^2}$$

$$\Rightarrow [g] = 1$$

such that

$$[R] = [\partial]^2 = L^{-2} = \mu^2$$

4. From (7.8) it follows $[G]^{-1} [dx]^4 [R] = [G]^{-1} \mu^{-4} \mu^2 = [S] \overset{!}{=} 1$

$$\Rightarrow \boxed{\log_\mu [G] = -2 < 0}$$

Consistency:

$$G = \frac{\hbar c}{m_p^2} = \frac{1}{m_p^2} \quad ✓$$

→ Einstein gravity is superficially non-renormalizable!