7. Systematics of Renormalization

Remember:

- **IR-divergences**:
  - Due to massless particles (photons)
  - Regulate with small photon mass $\mu$
  - Divergences from soft virtual photons (vertex correction)
    and soft bremsstrahlung cancel

- **UV-divergences**:
  - Due to unbounded high momenta of particles (= unbounded small length scales) in all three radiative corrections:
    - Regulate with additional heavy particles ($\Lambda$) or dimensional regularization ($\xi$)
    - Cancelled in some observable quantities
    - Diverging differences between physical and bare quantities

→ Fundamental problem → Study UV-divergences systematically
7.1 Counting UV-Divergences

1. Goal: Classify UV-divergences in QED

2. Definition:
   \[ N_e = \# \text{ external } e^- \text{ lines} \]
   \[ N_N = \# \text{ external } N^- \text{ lines} \]
   \[ P_e = \# \text{ } e^- \text{ propagators} \]
   \[ P_N = \# \text{ } N^- \text{ propagators} \]
   \[ V = \# \text{ vertices} \]
   \[ L = \# \text{ independent loops} \]

3. Superficial degree of divergence:

   \[ D_{QED} = (3L + V) - (P_e + 2P_N) \]

   \[ D_{QED} \begin{cases} > 0 : & \text{Diverges with } \Lambda^D \\ = 0 : & \text{Divergence with } \log \Lambda \\ < 0 : & \text{No divergence} \end{cases} \]
Example:

\[ \sim \log \Lambda \quad \text{and} \quad \mathcal{D}_{\text{QED}} = 4.1 - (2 + 2.1) = 0 \]

However: Not always correct!

Reasons:

- Divergence may be weaker (or ascent) if symmetries make divergent terms cancel:

\[ \sim \log \Lambda \quad \text{although} \quad \mathcal{D} = 4.1 - (2 + 2.1) = 2 \]

- Divergence may be worse if diagram contains divergent subdiagram:

\[ \sim \log \Lambda \quad \text{although} \quad \mathcal{D} = 4.1 - (2 + 2.2) = - \]

Susdiagram diverges
Tree-level diagrams without propagators have $D=0$ but no divergence:

\[ D = 4.0 - (0 + 2.0) = 0 \]

4. Use

\[ L = P_e + P_r - V + 1 = \text{Cycle space dimension} \quad (7.1) \]

\[ V = 2P_r + N_r = \frac{1}{2}(2P_e + N_e) \quad (7.2) \]

To show

\[ D_{\text{QED}} = 4 - N_r - \frac{3}{2} N_e \quad \rightarrow \text{independent of number of vertices!} \]

5. Aside: Fury's theorem

Feynman diagrams with an odd number of photons as their only external lines vanish identically.

Proof: Follows from charge conjugation symmetry (C) of QED.
6. Enumerate diagrams with $D_{	ext{QED}} > 0$:

a) $N_\sigma = 0$

i) $N_\tau = 0$ ($D = 4$) \[ \sim \text{badly divergent} \]

\[ \rightarrow \text{Unobservable vacuum energy shift} \rightarrow \text{ignore!} \]

ii) $N_\eta = 1$ ($D = 3$) \[ \text{Fully} \sim 0 \]

iii) $N_\eta = 2$ ($D = 2$); Recall our first-order result (6.36):

\[ (g_{\mu \nu} g^{\mu \nu} - g_{\mu \nu} g^{\mu \nu}) \cdot \Pi (q^2) \]

\[ \sim (g_{\mu \nu} q^\mu q^\nu) \cdot \text{some form} \sim (g_{\mu \nu} q^\mu q^\nu) \cdot \text{log} \]

\[ \sim (g_{\mu \nu} q^\mu q^\nu) \cdot \text{log} \]

\[ \sim (g_{\mu \nu} q^\mu q^\nu) \cdot \text{some form} \sim (g_{\mu \nu} q^\mu q^\nu) \cdot \text{log} \]

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iv.) $N_n = 3$ ($D = 1$)

\[
F_{mm} = 0
\]

v.) $N_n = 4$ ($D = 0$)

\[
Ward \sim 1
\]

**Notes:**

- Light-by-light scattering (Halpern scattering)
  \[
  \sigma_{LL} \sim \left(\frac{\alpha^2}{\mu_F^4}\right)^2
  \]
- Lowest order:
  \[
  \sim \frac{\alpha^2}{\mu_F^4}
  \]
b) \( N_e = 2 \):

i.) \( N_e = 0 \) (\( D = 1 \)); Recall our first-order result in (6.28):

\[
\sim \text{const.} \log \Lambda + \frac{e}{4 \pi} \text{const.} \log \Lambda
\]

\( \alpha_n (\Lambda) \)

\( \alpha_2 (\Lambda) \)

ii.) \( N_e = 1 \) (\( D = 0 \)); Recall our first-order result in (6.12):

\[
\sim -i e \mathcal{M} \log \Lambda
\]

\( \alpha_3 (\Lambda) \)

\( \rightarrow \) Diagrams only diverge if they contain (7.3), (7.4), or (7.5) as subdiagram

\( \rightarrow \) QED only contains four UV-divergent numbers: \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \)
7. Idea: Absorb finite number of diverging quantities in finite number of diverging
but unobservable Lagrangian parameters \( \rightarrow \) Renormalization (\( \Rightarrow \) next lecture)

8. Generalization: \( \mathcal{L} \) QED in \( d \) spacetime dimensions \( \rightarrow \)

\[
\mathcal{L}_{\text{QED}} = dL - Pe - 2Pr
\]
\[
\begin{cases} \ \ \ \ \text{(8.1)} & \text{\( \Rightarrow \) (8.2)} \\ o \ = d + \left( \frac{d-4}{2} \right) V - \left( \frac{d-2}{2} \right) N_e - \left( \frac{d-1}{2} \right) N_q \end{cases}
\]

\( \rightarrow \) Observations:

- \( d < 4 \), diagrams of higher order \( (V \rightarrow \infty) \) are always superficially convergent
- \( d = 4 \), \( \mathcal{L}_{\text{QED}} \) is independent of the order of \( V \)
- \( d > 4 \), diagrams of higher order \( (V \rightarrow \infty) \) are always superficially divergent
9. **Super-Nonrenormalizable theory:**

Only a finite number of Feynman diagrams (!) superficially diverge.

Example: QED in $d=2+1$

**Renormalizable theory:**

Only a finite number of amplitudes (!) superficially diverge.

→ Divergence at all orders in perturbation theory

Example: QED in $d=3+1$

**Non-Renormalizable theory:**

All amplitudes diverge at sufficiently high orders in perturbation theory.

Example: QED in $d=4+1$
Alternative approach:

1. $\phi^4$-theory:
   \[ Z_{\phi^4} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \quad n \in \mathbb{N} \quad (7.6) \]

2. Definitions:
   \[
   \begin{aligned}
   N_\phi &= \# \text{ external lines} \\
   P_\phi &= \# \text{ propagators} \\
   V &= \# \text{ vertices} \\
   L &= \# \text{ independent loops}
   \end{aligned}
   \]

3. Superficial degree of divergence:
   \[
   \Delta \phi^4 \equiv dL - 2P_\phi \\
   = d + \left[ n \left( \frac{d-2}{2} \right) - d \right] \cdot V - \left( \frac{d-2}{2} \right) N_\phi \quad \text{(7.7)}
   \]

   - For $n=4$ in $d=4$, independent of $V \rightarrow$ renormalizable
4. Alternative approach via dimensional analysis:

a) Recall \( t = c = 1 \) and \( \lambda_c = \frac{t}{mc} = \frac{2\pi}{m} \)

→ Dimension of length \( [\lambda_c] = M^{-1} \) (\( M \): dimension of mass)

b) \( [S] = 1 \) since \( t = 1 \)

c) \( S = \int d^d x \) and \( [d^d x] = M^{-d} \)

→ \( [x] = M^d \)

d) From (7.6) and using \( [S] = M \):

\[
[\phi] = M^{\frac{d-2}{2}}
\]

\[
[w] = M \quad \text{(consistent!)}
\]

\[
[\lambda] = M^{d-1} (d-2)/2
\]
c) Amplitude $\mathcal{M}$ of single diagram with $N\phi$ external lines

- Could arise (on tree-level) from interaction $\phi^N \phi \rightarrow [\phi] = \mathcal{M}$ $d-N\phi (d-2)/2$

$\rightarrow [\mathcal{M}] = [\phi] = M$

Recall: $\mathcal{M} = -\lambda + O(\lambda^2)$ from Eq. (4.16)

d) $\mathcal{M} \rightarrow M \sim \lambda^D$ for UV-cutoff $\lambda \rightarrow \infty$

$\mathcal{M} = \lambda$

$[\lambda]^V [\lambda]^D = [\mathcal{M}] = M$

$\Rightarrow V \log \mu [\lambda] + D = d-N\phi (d-2)/2$

"mass dimension of $[\lambda]"$

$D\phi = d - \frac{\log \mu [\lambda]}{\mu} V - \left(\frac{d-2}{2}\right) N\phi = (7.2)$
5. Equivalent characterization:

- **Super-renormalizable theory:**
  
  Coupling constant has positive mass dimension: \( \log[\lambda] > 0 \)

- **Renormalizable theory:**
  
  Coupling constant is dimensionless: \( \log[\lambda] = 0 \)

  Example: QED with \( [e] = 1 \) is superficially renormalizable.

- **Non-renormalizable theory:**
  
  Coupling constant has negative mass dimension: \( \log[\lambda] < 0 \)
Aside: Why quantum gravity is special

1. Fields: Components of the metric tensor $g_{\mu\nu}(x)$

2. Einstein-Hilbert action of pure gravity:

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{|\det g(x)|} \left[ R(g(x)) - 2\Lambda_c \right]$$

$R = g^{\mu\nu} R_{\mu\nu}$: Ricci scalar with Ricci tensor $R_{\mu\nu}$

$\Lambda_c$: Cosmological constant

$G$: Gravitational constant = Coupling constant of gravity

$\Rightarrow$ Equations of motion = Einstein's field equations (in vacuum):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_c g_{\mu\nu} = 0$$
3. Recall: \( R \sim g^{\mu\nu} R_{\mu\nu} \sim g^{\mu\nu} R^\gamma_{\mu\nu \alpha} \sim \frac{1}{g} \sum \delta_{\alpha} \) 
   \( \Rightarrow \) \([R] = [g]^3 \) $[g] \geq 2$
   \[ ] \quad \text{Christoffel symbol (of 2nd kind)}
   \[ ] \quad \text{Riemann curvature tensor}

and \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \)

\[ \Rightarrow \mathcal{L} = [ds^2] = [g] \int [dx]^2 = [g] \) $[g] \geq 2$

\[ \Rightarrow \) $[g] = 1$

such that

\[ [R] = [g]^2 = \mathcal{L}^{-2} = \mathcal{L}^2 \]

4. From (7.8) follows \( [G]^{-1} [dx]^4 [R] = [G]^{-4} \mathcal{M}^{-4} \mathcal{M}^2 = \) $[S] = 1$

\[ \Rightarrow \log \mu [G] = -2 < 0 \]

\[ \quad \text{Constraint:} \quad G_i = \frac{H \pm c}{m_p} - \frac{1}{m_p^2} \quad \checkmark \]

\[ \Rightarrow \text{Einstein gravity is superficially non-renormalizable}! \]