7.2. Renormalized Perturbation Theory

Goal: Compute finite predictions from given physical parameters m and e for $\Lambda \to \infty$

Recipe:

(i) Compute UV-divergent amplitude with UV-regulator Λ to some order in α_0:

$$\mathcal{M} = \mathcal{M}(m_0, e_0; \Lambda) + \mathcal{O}(\alpha_0^*)$$

(ii) Compute physical mass, physical charge and field-strength renormalization:

$$m = m(m_0, e_0; \Lambda) + \mathcal{O} \quad , \quad e = e(m_0, e_0; \Lambda) \quad , \quad Z = Z(m_0, e_0; \Lambda) + \mathcal{O}$$

(iii) Renormalization.

Eliminate m_0 and e_0 in favour of m and e (fixed and given by experiments):

$$e_0 = e_0(m, e; \Lambda) \quad \text{and} \quad m_0 = m_0(m, e; \Lambda)$$

(iv) Then $$\mathcal{M}(m, e) \equiv \lim_{\Lambda \to \infty} \mathcal{M}(m_0(m, e; \Lambda), e_0(m, e; \Lambda); \Lambda)$$

is finite and independent of Λ in all orders of α.

→ Bare perturbation theory

→ Works for all renormalizable QFTs

→ Alternative (but equivalent!) formalism: Renormalized perturbation theory

1. Ex ϕ^4-theory in $d = 3+1$ dimension:

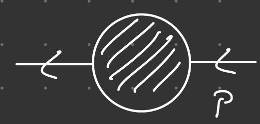
$$\mathcal{L}_{\phi^4} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!}\phi^4$$

2. With $D_{\phi^4} = 4 - N_\phi$ ($\Rightarrow$ (7.7)) and $N_\phi = 0, 2, 4, \dots$ one finds the divergent amplitudes:

$D_{\phi^4} = 4$

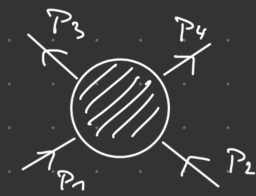
unobservable vacuum energy shift

$D_{\phi^4} = 2$



$\sim \Lambda^2 + \log \Lambda \, p^2$

$D_{\phi^4} = 0$



$\sim \log \Lambda$

→ 3 divergent quantities

→ Absorb in 3 unobservable parameters: bare mass m_0, bare coupling λ_0, fields ϕ

3. <u>Recall:</u>

$$\int d^4x\, e^{ipx} \langle \Omega | T\, \phi(x)\, \phi(0) | \Omega \rangle = \frac{i Z}{p^2 - m^2} + \ldots\ldots$$

Absorb unobservable Z in rescaled fields:

$$\boxed{\phi_r \equiv \frac{1}{\sqrt{Z}}\, \phi}$$

Then

$$\int d^4x\, e^{ipx} \langle \Omega | T\, \phi_r(x)\, \phi_r(0) | \Omega \rangle = \frac{i \cdot 1}{p^2 - m^2} + \ldots\ldots$$

4. → Lagrangian in new fields:

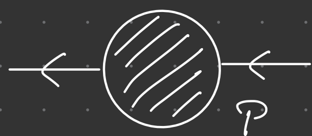
$$\mathcal{L}_{\phi^4} = \frac{1}{2} Z(\partial_\mu \phi_r)^2 - \frac{1}{2} m_0^2 Z \phi_r^2 - \frac{\lambda_0}{4!} Z^2 \phi_r^4$$

5. <u>Split</u> terms into observable parameters and unobservable ones:

$$\begin{aligned}\mathcal{L}_{\phi^4} = {} & \overbrace{\frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!} \phi_r^4}^{\text{Physical parameters (fixed)}} \\ & \underbrace{+ \frac{1}{2}\underbrace{(Z-1)}_{\delta_Z}(\partial_\mu \phi_r)^2 - \frac{1}{2}\underbrace{(m_0^2 Z - m^2)}_{\delta_m}\phi_r^2 - \frac{1}{4!}\underbrace{(\lambda_0 Z^2 - \lambda)}_{\delta_\lambda}\phi_r^4}_{\text{Counterterms (cutoff-dependent)}}\end{aligned} \tag{7.9}$$

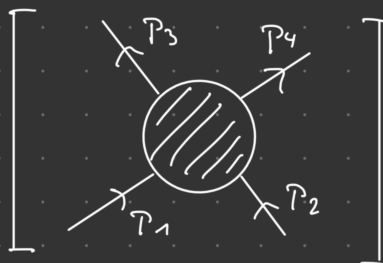
→ δ_Z, δ_m, and δ_λ absorb unobservable, diverging shifts of bare and physical quantities

6. Experimental input $\longrightarrow$ <u>Renormalization conditions:</u>



2 conditions

$$\overset{!}{=} \frac{i}{p^2 - m^2} + \ldots \qquad (7.10)$$



1 condition

$$\overset{!}{=} -i\lambda \qquad (7.11)$$

fc & a
$p_i = (m, \vec{0})$

Note: Recall: $i\mathcal{M}(p_1 p_2 \mapsto p_3 p_4) = -i\lambda_0 + O(\lambda_0^2)$

7. Perturbation theory of (7.9) $\longrightarrow$

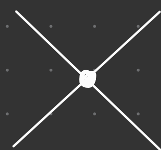
Feynman rules for renormalized perturbation theory of ϕ^4-theory in momentum space for S-matrix elements:

1. Edges:

$$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

↑ physical mass!

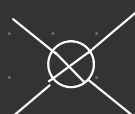
2. Vertices:



$$= -i\lambda$$

↑ physical coupling!

Counterterms



$$= -i\delta_\lambda$$

$$= i(p^2\delta_Z - \delta_m)$$

3. External lines

$$= 1$$

4. Impose momentum conservation at all vertices

5. Integrate over all undetermined momenta

6. Divide by the symmetry factor

8. Procedure for computing amplitudes:

a) Sum all relevant diagrams built from the Feynman rules above

b) If loop integrals diverge, introduce a regulator

c) The results depend on the (yet undetermined) parameters $\{\delta_\bullet\}$, the fixed physical parameters m and e, and the regulator (Λ or ε)

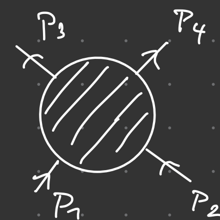
d) Choose ("renormalize") the parameters $\{\delta_\bullet\}$ such that the renormalization conditions (7.10) and (7.11) are satisfied.

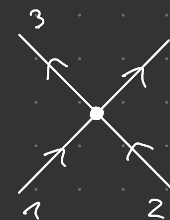
e) With these $\{\delta_\bullet\}$, the amplitude is finite, independent of the regulator, and depends only on the physical parameters.

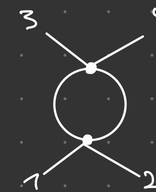
9. Bare perturbation theory and renormalized perturbation theory are equivalent and yield the same results.

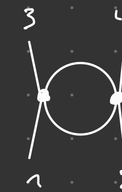
10. Example in one-loop order:

a) X Amplitude

$\mathcal{M}(p_1 p_2 \mapsto p_3 p_4) =$ 

$=$ 

$+$ 

$+$ 

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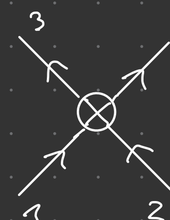
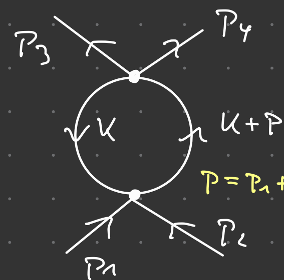
$+$ 

$$\overset{\circ}{=} -i\lambda + (-i\lambda)^2 \left[iV(s) + iV(t) + iV(u) \right] - i\delta_\lambda$$

of order λ^2 (see below)

with Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_3 - p_1)^2$, $u = (p_4 - p_1)^2$

b) Evaluate loop integral with dimensional regularization:

$(-i\lambda)^2 \cdot iV(s) =$  $= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(k+p)^2-m^2}$

$p = p_1 + p_2 \Rightarrow p^2 = s$

Feynman parameter, Substitution, Wick rotation, Dimensional regularization

$$\overset{\varepsilon\to 0}{\sim} -(-i\lambda)^2 \frac{i}{32\pi^2} \int_0^1 dx \left\{ \frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log\left[m^2 - x(1-x)p^2\right] \right\}$$

c) Enforce renormalization condition (7.11) to determine δ_λ:

$$i\mathcal{M}\Big|_{\substack{s=4m^2 \\ t=u=0}} \overset{!}{=} -i\lambda$$

solved by

$$\delta_\lambda := -\lambda^2\left[V(4m^2) + 2V(0)\right] \overset{\varepsilon\to 0}{\sim} \frac{\lambda^2}{32\pi^2}\int_0^1 dx \left\{ \frac{6}{\varepsilon} - 3\gamma + 3\log(4\pi) - \log\left[m^2 - x(1-x)4m^2\right] - 2\log\left[m^2\right] \right\}$$

d) $\to$ Amplitude

$$i\mathcal{M}(p_1p_2 \to p_3p_4) = -i\lambda - i\lambda^2 \cdot \tilde{\mathcal{F}}(\{p_i\}; m)$$

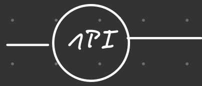
$\mathcal{F}$: finite function of the momenta $\{p_i\}$, parametrized by the physical mass m.

Important: the regulator ε drops out!

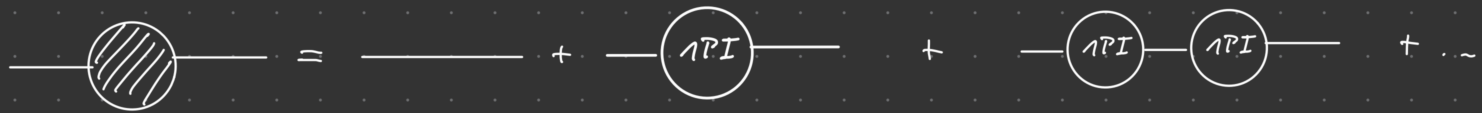
e) Enforce renormalization condition (7.10) to determine δ_Z and δ_m:

i) Define

$$\boxed{-i\mathcal{M}^2(p^2) := \text{—1PI—}}$$



ii) It follows

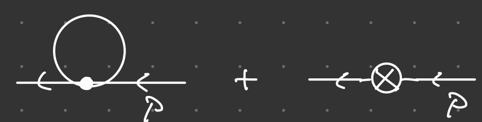


$$= \frac{i}{p^2-m^2-M^2(p^2)} \overset{!}{=} \frac{i\cdot 1}{p^2-m^2} + \dots$$

iii) $\rightarrow$ (7.10) is equivalent to

$$\boxed{ M^2(p^2)\Big|_{p^2=m^2} \overset{!}{=} 0 \quad \text{and} \quad \frac{dM^2(p^2)}{dp^2}\Big|_{p^2=m^2} \overset{!}{=} 0 } \qquad (7.12)$$

iv) In one-loop order: $-iM^2(p^2) \approx$



$$= (-i\lambda)\cdot\frac{1}{2}\cdot\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2} + i(p^2\delta_Z-\delta_m)$$

$$= (-i\lambda)\cdot\frac{1}{2}\cdot\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2}\;+\;i\left(p^2\delta_Z-\delta_m\right)$$

Wick rotation, Dimensional regularization

$$\overset{\circ}{=} -\frac{i\lambda}{2}\,\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}}\;+\;i\left(p^2\delta_Z-\delta_m\right)$$

$\rightarrow$ (7.12) solved by

$$\delta_Z := 0 \quad\text{and}\quad \delta_m := -\frac{\lambda}{2}\,\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}}$$

Note: $\delta_Z=0$ is a special case of ϕ^4-theory in one-loop order

Application to QED:

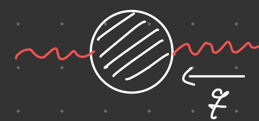
1. Original Lagrangian:

$$\mathcal{L}_{QED} = \frac{1}{4}\left(F_{\mu\nu}\right)^2 + \bar{\psi}\left(i\not{\partial} - m_0\right)\psi - e_0\, \bar{\psi}\gamma^\mu\psi A_\mu$$

2. Interacting propagators:

$$= \frac{i Z_2}{\not{p} - m} + \dots$$

and



$$= \frac{-i Z_3\, g_{\mu\nu}}{q^2} + \dots$$

3. $\to$ Renormalized fields:

$$\psi_r := \frac{1}{\sqrt{Z_2}}\psi \qquad \text{and} \qquad A_r^\mu := \frac{1}{\sqrt{Z_3}} A^\mu$$

4. $\to$

$$\mathcal{L}_{QED} = -\frac{1}{4} Z_3 \left(F_r^{\mu\nu}\right)^2 + Z_2\, \bar{\psi}_r\left(i\not{\partial} - m_0\right)\psi_r - e_0 Z_2 Z_3^{1/2}\, \bar{\psi}_r \gamma^\mu \psi_r (A_r)_\mu$$

5. $\to$ $Z_1 := Z_2 Z_3^{1/2} \cdot \frac{e_0}{e}$

6. $\rightarrow$

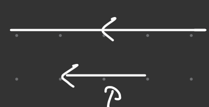
$$\mathcal{L}_{QED} = -\frac{1}{4}\left(F_r^{\mu\nu}\right)^2 + \overline{\psi}_r(i\not{\partial}-m)\psi_r - e\,\overline{\psi}_r\gamma^\mu\psi_r\,(A_r)_\mu$$

$$\underbrace{-\frac{1}{4}\delta_3\left(F_r^{\mu\nu}\right)^2 + \overline{\psi}_r\left(i\delta_2\not{\partial}-\delta_m\right)\psi_r - e\,\delta_1\,\overline{\psi}_r\gamma^\mu\psi_r\,(A_r)_\mu}_{\text{4 counterterms}}$$

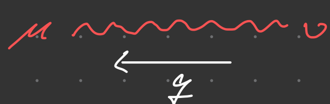
with $\delta_i := Z_i - 1$ for $i = 1,2,3$ and $\delta_m := Z_2 m_0 - m$

7. $\rightarrow$ Feynman rules (only propagators & vertices):

Edges:

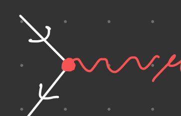


$$= \frac{i}{\not{p}-m+i\varepsilon}$$



$$= \frac{-i g_{\mu\nu}}{q^2+i\varepsilon}$$

Vertices:



$$= -ie\gamma^\mu$$

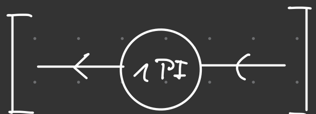
$$= -ie\delta_1\gamma^\mu$$

$$= i(\not{p}\,\delta_2 - \delta_m)$$

$$= -i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\delta_3$$

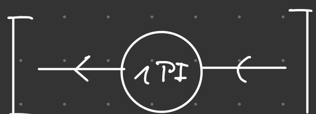
8. 4 counterterm coefficients $\to$ 4 <u>renormalization conditions</u>

1. Fix electron mass to m:



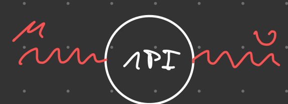
$$\Big[\ \cdots\ \Big]_{\not p = m} = -i\,\Sigma(\not p = m) \overset{!}{=} 0$$

2. Fix residue of electron propagator to 1 (choose ψ_r):



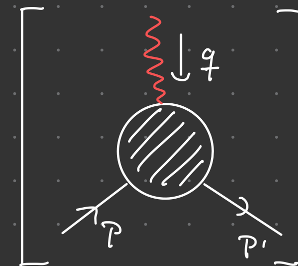
$$\frac{d}{d\not p}\Big[\ \cdots\ \Big]_{\not p = m} = -i\,\frac{d\Sigma(\not p)}{d\not p}\bigg|_{\not p = m} \overset{!}{=} 0$$

3. Fix residue of photon propagator to 1 (choose A_r):



$$\left[\frac{\ \cdots\ }{(g^{\mu\nu}q^2 - g^{\mu}q^{\nu})}\right]_{q^2=0} = i\Pi(q^2=0) \overset{!}{=} 0$$

4. Fix electron charge to e:



$q = p' - p$

$$\Big[\ \cdots\ \Big]_{\text{fc}\ \&\ q=0} = -ie\,\Gamma^{\mu}(q=0) \overset{!}{=} -ie\gamma^{\mu}$$