8. Functional Methods

So far:

Hamiltonian ⟶ Canonical quantization ⟶ Feynman rules

Alternative:

Lagrangian ⟶ Path integral ⟶ Feynman rules

- Two descriptions of the same physics!
- Application: Derivation of the photon propagator (easier with path integrals)

8.1. Path Integrals in Quantum Mechanics

1. Nonrelativistic particle in 1D: $H = \frac{p^2}{2m} + V(x)$

2. Time evolution operator (from canonical quantization): $U(x_a, x_b; T) = \langle x_b | e^{-\frac{i}{\hbar} HT} | x_a \rangle$

3. Path integral formalism ⟶ alternative expression for U:

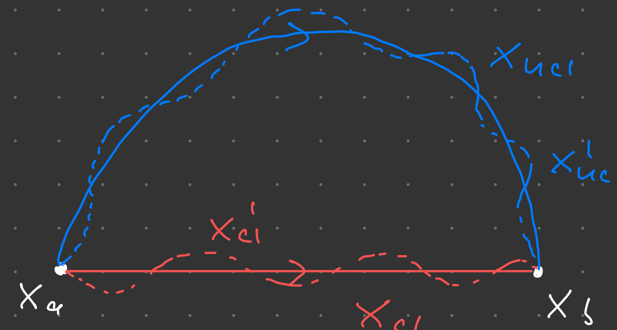
$$U(x_a, x_b; T) = \underbrace{\sum_{\text{all paths } x(t)}}_{\text{Superposition principle}} \underbrace{e^{i\overset{\text{Functional}}{\mathcal{F}}[x(t)]}}_{\text{Pure phase}} = \int \mathcal{D}x(t)\, e^{i\mathcal{F}[x(t)]}$$

4. Conditions on $\mathcal{F}$:

a) Describes the System

b) Functional of path $x(t)$

c) Classical path $x_{cl}(t)$ dominates (for $\hbar \to 0$) $\rightarrow$ $U(x_a, x_b; T) \approx \sum'_{\substack{\text{paths close} \\ \text{to } x_{cl}(t)}} e^{i\mathcal{F}[x(t)]}$



$$e^{i\mathcal{F}[x_{ucl}]} + e^{i\mathcal{F}[x'_{ucl}]} \approx 0$$

$$e^{i\mathcal{F}[x_{cl}]} + e^{i\mathcal{F}[x'_{cl}]} \neq 0$$

Therefore $\left.\frac{\delta \mathcal{F}}{\delta x}\right|_{x = x_{cl}} \overset{!}{=} 0 \Rightarrow \mathcal{F} = \frac{S}{\hbar} = \frac{1}{\hbar} \int dt\, L(x(t))$

5. Propagation amplitude (Propagator):

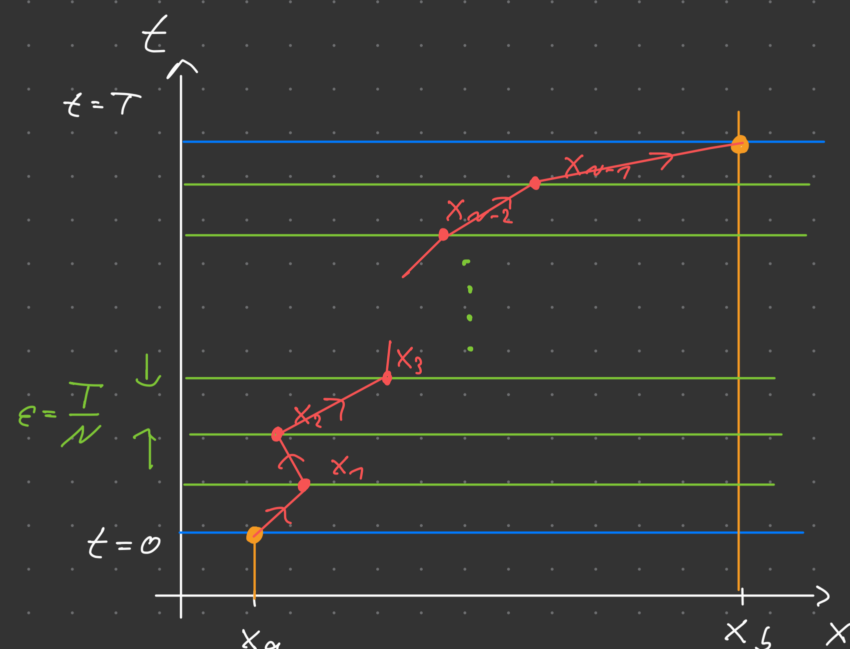
$$\boxed{U(x_a, x_b; T) = \int \mathcal{D}x(t)\, e^{\frac{i}{\hbar} S[x(t)]} \overset{?}{=} \langle x_b | e^{-\frac{i}{\hbar} HT} | x_a \rangle} \qquad (8.1)$$

6. Definition of PI via time slices:

$$\int \mathcal{D}x(t) := \lim_{N\to\infty} \frac{1}{C_\varepsilon} \int \frac{dx_1}{C_\varepsilon} \cdot \ldots \cdot \int \frac{dx_{N-1}}{C_\varepsilon}$$

$$= \lim_{N\to\infty} \frac{1}{C_\varepsilon} \prod_{k=1}^{N-1} \int \frac{dx_k}{C_\varepsilon}$$

with $\varepsilon = \frac{T}{N}$ and C_ε a constant (to be determined)



Example 8.1: Particle in potential $V(x)$

1. Lagrangian: $L = \frac{m}{2}\dot{x}^2 - V(x)$

2. Action:

$$S = \int_0^T dt\, L \approx \sum_{n=0}^{N-1} \left\{ \frac{m}{2} \frac{(x_{n+1}-x_n)^2}{\varepsilon} - \varepsilon V\left(\frac{x_{n+1}+x_n}{2}\right) \right\}$$

3. Recursion:

$$U(x_a, x_b; T) = \int_{-\infty}^{\infty} \frac{dx'}{C_\varepsilon} \exp\left[\frac{i}{\hbar} \frac{m(x_b-x')^2}{2\varepsilon} - \frac{i}{\hbar}\varepsilon V\left(\frac{x_b+x'}{2}\right)\right] \times U(x_a, x'; T-\varepsilon)$$

$$= \int_{-\infty}^{\infty} \frac{dx'}{C_\varepsilon} \exp\left[\frac{i}{\hbar}\frac{m(x_b-x')^2}{2\varepsilon}\right] \times \left[1 - \frac{i}{\hbar}\varepsilon V(x_b) + \dots\right] \times \left[1 + (x'-x_b)\frac{\partial}{\partial x_b} + \frac{(x'-x_b)^2}{2}\frac{\partial^2}{\partial x_b^2} + \dots\right] \times U(x_a, x_b; T-\varepsilon)$$

$$\overset{\circ}{=} \overbrace{\frac{1}{C_\varepsilon}\sqrt{\frac{2\pi\hbar\varepsilon}{-im}}}^{\overset{!}{=}1} \times \left[1 - \frac{i}{\hbar}\varepsilon V(x_b) + \frac{i\hbar}{2m}\varepsilon\frac{\partial^2}{\partial x_b^2} + O(\varepsilon^2)\right] U(x_a, x_b; T-\varepsilon)$$

4. PI measure:

$$\boxed{C_\varepsilon = \sqrt{\frac{2\pi\hbar\varepsilon}{-im}}}$$

(Not generic but depends on $\mathcal{L}$!)

5. Use $U(x_a, x_b; T-\varepsilon) = U(x_a, x_b, T) - \varepsilon\, \partial_T U + O(\varepsilon^2)$

and compare terms $\propto \varepsilon$:

$$\boxed{i\hbar\, \partial_T U = \underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2} + V(x_b)\right]}_{H} U = H U}$$

Schrödinger Equation

6. Initial condition: set $N=1$ $\longrightarrow$

$$U(x_a, x_b; \varepsilon) = \frac{1}{C_\varepsilon} \exp\left[\frac{i}{\hbar}\frac{m}{2\varepsilon}(x_b - x_a)^2 + O(\varepsilon)\right] \approx \sqrt{\frac{-im}{2\pi\hbar\varepsilon}}\, e^{\frac{i}{\hbar}\frac{m}{2\varepsilon}(x_b-x_a)^2}$$

$$\xrightarrow{\varepsilon\to 0} \delta(x_a - x_b) = U(x_a, x_b; 0) = \langle x_b | x_a \rangle$$

7. The last two steps conclude the proof of the second equality in (8.1) for

$$H = \frac{p^2}{2m} + V(x).$$

Generalization (⇒ P-Set 12)

1. ∄ Coordinates q_i, conjugate momenta p_i, Hamiltonian $H(\vec{q}, \vec{p})$

$\hbar = 1$

2. Canonical quantization: $[q_i, p_j] = i\hbar\,\delta_{ij} \rightarrow U(\vec{q}_a, \vec{q}_b; T) = \langle \vec{q}_b | e^{-i\hat{H}T} | \vec{q}_a \rangle$

3. Time slicing: $e^{-i\hat{H}T} = \underbrace{e^{-i\hat{H}\cdot\varepsilon} \cdots\cdots e^{-i\hat{H}\varepsilon}}_{\times N}$ with $\varepsilon = \frac{T}{N}$

4. Insert $N-1$ identities $\mathbb{1}_k = \int d\vec{q}_k \, |\vec{q}_k\rangle\langle\vec{q}_k|$ $(k = 1, \dots, N-1)$

$$\langle \vec{q}_{k+1} | e^{-i\hat{H}\varepsilon} | \vec{q}_k \rangle = \langle \vec{q}_{k+1} | \mathbb{1} - i\hat{H}\varepsilon + \mathcal{O}(\varepsilon^2) | \vec{q}_k \rangle$$

5. For $\hat{H} = \hat{H}_1(\vec{q}) + \hat{H}_2(\vec{p})$

$$\langle \vec{q}_{k+1} | \hat{H} | \vec{q}_k \rangle \overset{\circ}{=} \int \frac{d\vec{p}_k}{2\pi} \, H\left(\frac{\vec{q}_{k+1} + \vec{q}_k}{2}, \vec{p}_k \right) \exp\left[i\vec{p}_k \cdot (\vec{q}_{k+1} - \vec{q}_k) \right]$$

6. Hamiltonian phase-space path integral:

$$U(\vec{q}_a,\vec{q}_b;T) \hat{=} \underbrace{\int_{\vec{q}_a}^{\vec{q}_b} \mathcal{D}\vec{q}(t) \int \mathcal{D}\vec{p}(t)}_{\lim_{N\to\infty} \prod_n \int \frac{d\vec{q}_n \, d\vec{p}_n}{2\pi\hbar}} \exp\left[\frac{i}{\hbar} \overbrace{\int_0^T dt \left(\underbrace{\vec{p}\cdot\dot{\vec{q}} - H(\vec{q},\vec{p})}_{\hat{=} L(q,\dot{q})}\right)}^{\hat{=} S[q]}\right] \qquad (8.2)$$

Notes:

- $\vec{p}$ and $\vec{q}$ do <u>not</u> satisfy the Hamiltonian EOMs
- If H depends only quadratically on p, the $\int \mathcal{D}\vec{p}(t)$ – integral can be solved
  $\rightarrow$ PI over $q(t)$ only
- Here the PI-measure is the <u>canonical measure</u> that becomes system-dependent when doing the $\int \mathcal{D}p(t)$ – integral ($\Rightarrow$ C_ε above)
- The Hamiltonian phase-space PI is the <u>most general form</u> of a PI

8.2. Path Integrals for Scalar Fields

Identification: $q_i \leftrightarrow \phi(x)$

Example 8.2: Real scalar field

$$\langle \phi_b | e^{-i\hat{H}T} | \phi_a \rangle = \int_{\phi_a}^{\phi_b} \mathcal{D}\phi\, \mathcal{D}\pi \, \exp\left[\frac{i}{\hbar} \int_0^T d^4x \left(\pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right) \right]$$

Evaluate π-integration

$$= \int_{\phi_a}^{\phi_b} \mathcal{D}\phi \, \exp\left[\frac{i}{\hbar} \int_0^T d^4x \, \mathcal{L}(\phi, \partial_\mu\phi) \right] \qquad (8.3)$$

- Lagrangian: $\mathcal{L}(\phi, \partial_\mu\phi) = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)$
- Boundaries: $\phi(\vec{x}, 0) \equiv \phi_a(\vec{x})$ and $\phi(\vec{x}, T) \equiv \phi_b(\vec{x})$

- Abandon Hamiltonian formalism and use (8.3) to define the time evolution
- All symmetries of $\mathcal{L}$ are manifest in the IP formalism (in particular: Lorentz invariance!)

Recall: the Hamiltonian is not invariant under boosts!

- Goal: Derive correlation functions directly from PIs

Correlation functions:

1.

$$\langle\Omega|T\underbrace{\phi_H(x_1)\phi_H(x_2)}_{\text{Operators}}|\Omega\rangle \overset{?}{\longleftrightarrow} \int_{\phi(-T)=\phi_a}^{\phi(+T)=\phi_b}\mathcal{D}\phi\,\underbrace{\phi(x_1)\phi(x_2)}_{\text{Numbers}}\,e^{i\int_{-T}^{T}d^4x\,\mathcal{L}(\phi)} \qquad (8.4)$$

2.

$$\int\mathcal{D}\phi(x) = \int\mathcal{D}\phi_1(\vec{x})\int\mathcal{D}\phi_2(\vec{x})\int_{\phi(x_1^0,\vec{x})=\phi_1(\vec{x})}^{\phi(x_2^0,\vec{x})=\phi_2(\vec{x})}\mathcal{D}\phi$$

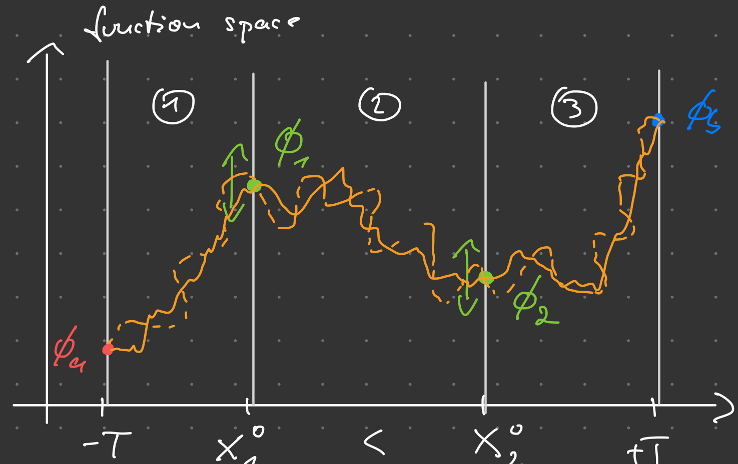
3.

$$(8.4) = \int\mathcal{D}\phi_1(\vec{x})\int\mathcal{D}\phi_2(\vec{x})\,\overbrace{\phi_1(\vec{x}_1)\,\phi_2(\vec{x}_2)}^{0}$$

$$\times\,\langle\phi_b|e^{-iH(T-x_2^0)}|\phi_2\rangle \quad 3$$

$$\times\,\langle\phi_2|e^{-iH(x_2^0-x_1^0)}|\phi_1\rangle \quad 2$$

$$\times\,\langle\phi_1|e^{-iH(x_1^0+T)}|\phi_a\rangle \quad 1$$



4. Use $\int \mathcal{D}\phi_1(\vec{x})\, |\phi_1\rangle\langle\phi_1| = \mathbb{1}$ and $\hat{\phi}_S(\vec{x}_1)|\phi_1\rangle = \phi_1(\vec{x})\,|\phi_1\rangle$ $\quad$ compare: $(\hat{q}|q\rangle = q|q\rangle)$

$$\textcircled{0}\times\textcircled{3}\times\textcircled{2}\times\textcircled{1} \overset{x_2^0 > x_1^0}{=} \langle\phi_3|\, e^{-iH(T-x_2^0)} \underbrace{\phi_S(\vec{x}_2)\, e^{-iH(x_2^0-x_1^0)}}_{\phi_H(x_2)} \underbrace{\phi_S(\vec{x}_1)\, e^{-iH(x_1^0+T)}}_{\phi_H(x_1)} |\phi_a\rangle$$

$$= \langle\phi_3|\, \underbrace{e^{-iHT}}_{\to |\Omega\rangle\langle\Omega|}\, \mathcal{T}\{\phi_H(x_1)\,\phi_H(x_2)\}\, \underbrace{e^{-iHT}}_{\to |\Omega\rangle\langle\Omega|} |\phi_a\rangle$$

$$\xrightarrow{T\to\infty(1-i\varepsilon)} C\cdot \langle\Omega|\,\mathcal{T}\,\phi_H(x_1)\,\phi_H(x_2)\,|\Omega\rangle$$

5. Result:

$$\boxed{\langle\Omega|\,\mathcal{T}\,\phi_H(x_1)\,\phi_H(x_2)\,|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\int \mathcal{D}\phi\; \phi(x_1)\,\phi(x_2)\, \exp\left[i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi)\right]}{\int \mathcal{D}\phi\; \exp\left[i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi)\right]}}$$