8.3. Application: Quantization of the Electromagnetic Field

Goal: Apply PI formalism to derive the photon propagator $\frac{-i g_{\mu\nu}}{k^2+i\varepsilon}$

1. Action: $S[A] = \int d^4x \left[-\frac{1}{4} (F_{\mu\nu})^2 \right]$

   Partial integration (assume $A \xrightarrow{x\to\infty} 0$), use $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

   $$\hat{=} \frac{1}{2} \int d^4x \, A_\mu(x) \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) A_\nu(x)$$

   Fourier transform

   $$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \underbrace{\tilde{A}_\mu(k) \left(-k^2 g^{\mu\nu} + k^\mu k^\nu \right) \tilde{A}_\nu(-k)}_{\square}$$

2. Set $\tilde{A}_\mu(k) = k_\mu \cdot \alpha(k) \longrightarrow \square = 0 \longrightarrow S[A] = 0 \longrightarrow \int \mathcal{D}A \, \overbrace{e^0}^{1} = \infty$ ↯ Problem

3. Problem: Gauge invariance $A_\mu \to A_\mu + \frac{1}{e} \partial_\mu \alpha$

   Integration over continuity of gauge-equivalent configurations $A_\mu \sim 0 \Leftrightarrow A_\mu \propto \partial_\mu \alpha$ leads to divergence!

4. Solution: Count each physical configuration once ($\Rightarrow$ Faddeev & Popov)

a) Gauge fixing: $G(A) \overset{!}{=} 0$ (e.g. Lorenz gauge: $G(A) = \partial_\mu A^\mu$)

b) Let $A^\alpha_\mu := A_\mu + \frac{1}{e}\partial_\mu \alpha$, then

$$1 = \int \mathcal{D}\alpha \, \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)$$

Note 8.1

$$1 = \left[\prod_i \int dg_i\right] \underbrace{\delta^{(n)}(\vec{g})}_{\delta(g_1)\cdot\delta(g_2)\cdot\ldots\cdot\delta(g_n)} \underset{g = g(\vec{a})}{=} \left[\prod_i \int da_i\right] \delta^{(n)}(\vec{g}(\vec{a})) \underbrace{\det}_{\text{Jacobian}}\underbrace{\left(\frac{\partial \vec{g}}{\partial \vec{a}}\right)}_{n\times n \text{ Jacobi matrix}}$$

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c) Assume that $\frac{\delta G(A^\alpha)}{\delta \alpha}$ is independent of A and α (true for the Lorenz gauge)

This cannot be satisfied for non-abelian gauge theories $\longrightarrow$ Ghost fields

d) $$\int \mathcal{D}A\, e^{iS[A]} = \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]}\, \delta(G(A^\alpha))$$

Substitute $\tilde{A} = A^\alpha = A + \frac{1}{e}\partial\alpha \longrightarrow \mathcal{D}\tilde{A} = \mathcal{D}A$

Use gauge invariance: $S[A] = S[\tilde{A}]$

$$= \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) \underbrace{\int \mathcal{D}\alpha}_{=\infty} \underbrace{\int \mathcal{D}\tilde{A}\, e^{iS[\tilde{A}]}\, \delta(G(\tilde{A}))}_{\text{only physically distinct configurations}} \qquad (8.6)$$

e) Choose $G(A) = \partial^\mu A_\mu - \omega(x) \longrightarrow \det\left(\frac{\delta G}{\delta \alpha}\right) = \det\left(\frac{1}{e}\partial^2\right)$

$$(8.6) = \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x)) \qquad (8.7)$$

f) True for any ω $\rightarrow$ True for <u>normalized linear combinations</u>:

$$(8.7) = \underbrace{\overset{\text{arbitrary constant}}{\mathcal{N}(\xi)}}_{\text{Normalization}} \int \underbrace{\mathcal{D}\omega\, e^{-i\int d^4x \frac{\omega^2}{2\xi}}}_{\text{Linear combination}} \left\{ \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x)) \right\}$$

$$= \mathcal{N}(\xi) \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]} \underbrace{\exp\left[-i\int d^4x \frac{(\partial^\mu A_\mu)^2}{2\xi}\right]}_{\text{New term (breaks gauge symmetry)}}$$

g) $\forall$ $O(\hat{A})$ <u>gauge invariant</u> operator: $O(\hat{A}^\alpha) = O(\hat{A})$

$$\boxed{\langle\Omega|T\, O(\hat{A})|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\int \mathcal{D}A\, O(A) \exp\left\{i\int_{-T}^{T} d^4x \left[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]\right\}}{\int \mathcal{D}A \exp\left\{i\int_{-T}^{T} d^4x \left[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]\right\}}}$$

5. New action:

$$\tilde{S}[A] = \int d^4x \left[-\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \right]$$

Partial integration & Fourier transform

$$\hat{=} \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \left[-k^2 g^{\mu\nu} + (1 - \underbrace{\xi^{-1}}_{\text{New!}}) k^\mu k^\nu \right] \tilde{A}_\nu(-k)$$

$\rightarrow$ Argument of Step 2 no longer applies!

6. Propagator

$$D_F^{\mu\nu}(x-y) = \langle \Omega | T A^\mu(x) A^\nu(y) | \Omega \rangle$$

$\rightarrow \langle \Omega | \tilde{A}^\mu(k) \tilde{A}^\nu(q) | \Omega \rangle = 0 \quad \text{for} \quad k \neq -q$

Therefore

$$\tilde{D}_F^{\mu\nu}(q) = \langle\Omega|\tilde{A}^\mu(q)\tilde{A}^\nu(-q)|\Omega\rangle$$

Use $\tilde{A}^\nu(-q) = (\tilde{A}^\nu(q))^*$ since A^ν is real

Add $+i\varepsilon$ for regularization to the action $\quad M^{\mu\nu}(k)$ (symmetric)

$$= \frac{\int \mathcal{D}A\, \tilde{A}^\mu(q)\tilde{A}^\nu(-q) \exp\left\{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \overbrace{\left[-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right]}^{M^{\mu\nu}(k)} \tilde{A}_\nu(-k)\right\}}{\int \mathcal{D}A \exp\left\{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k)\left[-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right]\tilde{A}_\nu(-k)\right\}}$$

PI measure: $\mathcal{D}A = \prod_{\mu;\, k,\, k^0>0} d(\mathrm{Re}\,\tilde{A}^\mu(k))\, d(\mathrm{Im}\,\tilde{A}^\mu(k))$

Diagonalize $M^{\mu\nu}$, complete the square, and evaluate Gaussian integrals

(⇒ P-Set 12)

$$= i\left(M^{-1}(q)\right)^{\mu\nu}$$

Finally

$$\boxed{\tilde{D}_F^{\mu\nu}(q) \overset{\circ}{=} \frac{-i}{q^2+i\varepsilon}\left[g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}\right]}$$

7. Gauges:

* Set $\xi = 0$: $\tilde{D}_F^{\mu\nu}(q) = \frac{-i}{q^2+i\varepsilon}\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right]$ (Landau gauge)

* Set $\xi = 1$:

$$\boxed{\tilde{D}_F^{\mu\nu}(q) = \frac{-i g^{\mu\nu}}{q^2+i\varepsilon} \qquad \text{(Feynman gauge)}}$$

This form is Lorentz invariant!

Note 8.2

- Correlators of gauge invariant operators are independent of ξ.
- For $\xi \to \infty$ we have $M^{\mu\nu}(k) = -k^2 g^{\mu\nu} + k^\mu k^\nu$

  Since $(-k^2 g^{\mu\nu} + k^\mu k^\nu)\, k_\nu = 0$, the inverse $M^{-1}(k)$ does not exist!