9. Non-Abelian Gauge Theories

Motivation:

- So far: only ϕ^4 and $\bar{\psi}\gamma^\mu\psi A_\mu$ interactions
- Goal: Construct interacting theories of vector particles: A^4 or $(\partial A)A^2$?
- Problem: Quantization of vector fields is complicated by negative-norm states of the time component A^0:

$$[A_\mu(\vec{x}), \Pi_\nu(\vec{y})] = i\, g_{\mu\nu}\, \delta^{(3)}(\vec{x}-\vec{y}) \quad \text{but } g_{00} = -g_{ii}$$

- Observation: In Maxwell theory, negative-norm states are cancelled by longitudinal polarization states. This is rooted in the gauge symmetry.
- Idea: Generalize Maxwell theory (or QED, if matter is involved) to gauge theories with other symmetry groups.
- Spoiler: This type of theory turns out to describe all fundamental forces of nature (except gravity).

9.1. The Geometry of Gauge Invariance

1. $\measuredangle$ Local $U(1)$ symmetry G of Dirac field

$$\tilde{\psi}(x) = e^{i\alpha(x)}\psi(x) \qquad \text{for arbitrary } \alpha: \mathbb{R}^{1,3} \to \mathbb{R}$$

2. <u>Goal:</u> Construct invariant Lagrangian

3. No problem without derivatives:
   All terms invariant under global $U(1)$ transformations allowed (e.g. $\bar{\psi}(x)\psi(x)$)

4. $\measuredangle$ Directional derivative along $u \in \mathbb{R}^{1,3}$:

$$u^\mu \partial_\mu \psi := \lim_{\varepsilon\to 0} \frac{\psi(x+\varepsilon u) - \psi(x)}{\varepsilon}$$

$\psi(x+\varepsilon u)$ and $\psi(x)$ transform differently under G

$\rightarrow$ $u^\mu\partial_\mu\psi$ has no simple transformation law

($u^\mu\partial_\mu\tilde{\psi} \neq e^{i\alpha(x)} u^\mu\partial_\mu\psi$ but there are additional terms)

5. Postulate the existence of a "comparator" $U: \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \to \mathbb{C}$ with transformation

$$\tilde{U}(y,x) = e^{i\alpha(y)}\, U(y,x)\, e^{-i\alpha(x)} \quad \text{and} \quad U(y,y) = 1 \qquad (9.1)$$

[we require $U(y,x) = e^{i\phi(y,x)}$]

$\to$ $\psi(y)$ and $U(y,x)\cdot\psi(x)$ have same transformation law

6. Covariant derivative:

$$\boxed{u^\mu D_\mu \psi := \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon u) - U(x+\varepsilon u, x)\,\psi(x)}{\varepsilon}} \qquad (9.2)$$

7. Assume $U(y,x)$ continuous $\longrightarrow$

$$U(x+\varepsilon u, x) = 1 - ie\,\varepsilon u^\mu A_\mu(x) + O(\varepsilon^2) \qquad (9.3)$$

e: arbitrary constant

A_μ: new vector field = (gauge) connection

8. (9.3) in (9.2) $\xrightarrow{\circ}$

$$\boxed{D_\mu \psi(x) = \partial_\mu \psi(x) + ie A_\mu(x)}$$

9. (9.3) in (9.1) $\xrightarrow{\circ}$

$$\boxed{\tilde{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)}$$

10. $\xrightarrow{\circ}$

$$\boxed{\tilde{D}_\mu \tilde{\psi}(x) = e^{i\alpha(x)} D_\mu \psi(x)}$$

→ $D\psi$ transforms like ψ

→ All terms invariant under global $U(1)$ transformations allowed if ∂ is replaced by D

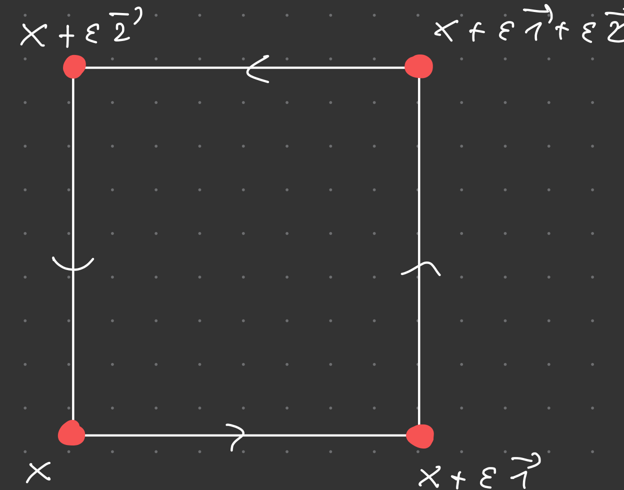
[e.g. $\bar{\psi}(x)\, i \not{D} \psi(x)$]

11. <u>Conclusion</u>:

Local symmetry → Gauge field A_μ needed for covariant derivatives

12. Kinetic energy term for A_μ? (only A_μ and its derivatives)

a) ∡ Locally invariant loop



$$\mathbb{U}(x) := U(x, x + \varepsilon \vec{2})$$

$$\times\, U(x + \varepsilon \vec{2}, x + \varepsilon \vec{1} + \varepsilon \vec{2})$$

$$\times\, U(x + \varepsilon \vec{1} + \varepsilon \vec{2}, x + \varepsilon \vec{1})$$

$$\times\, U(x + \varepsilon \vec{1}, x)$$

$\rightarrow$ $\tilde{\mathbb{U}} = \mathbb{U}$ by construction

b) $\rightarrow$ $\mathbb{U}(x) \overset{\circ}{=} 1 - i \varepsilon^2 e \underbrace{\left[\partial_1 A_2(x) - \partial_2 A_1(x)\right]}_{=: F_{12}} + \mathcal{O}(\varepsilon^3)$

$\rightarrow$ $\boxed{F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{(Field-strength tensor)}}$

is locally invariant by construction

13. Most general gauge invariant Lagrangian $D = 3+1$:

- Gauge invariance $\rightarrow$ Constructed from $\psi, D\psi, F_{\mu\nu}, \partial F_{\mu\nu}$ etc. and globally $U(1)$-invariant
- Relativistic $\rightarrow$ Lorentz scalar
- Renormalizable $\rightarrow$ Terms of mass dimension at most 4

$$\boxed{\mathcal{L} = \bar{\psi}(i\not{D})\psi - m\bar{\psi}\psi - \frac{1}{4}(F_{\mu\nu})^2 - \overbrace{c_1 \underbrace{\varepsilon^{\alpha\beta\mu\nu}}_{\text{pseudo tensor}} F_{\alpha\beta}F_{\mu\nu}}^{\text{pseudo scalar}} + \underbrace{c_2(\bar{\psi}\psi)^2 + \ldots}_{\text{non-renormalizable}}}$$

$\rightarrow$ Most general P/T-symmetric Lagrangian: Maxwell-Dirac $\rightarrow$ QED

9.2. The Yang-Mills Lagrangian

Goal: Replace local symmetry group $U(1)$ by non-abelian Lie group G

Examples: $O(3)$, $SU(2)$, $SU(3)$, ...

$\Rightarrow$ P-Set 13

1. $\exists$ Lie Group G represented by $n \times n$ unitary matrices V
   (Example: $G = SU(2)$, $V = 2\times 2$ unitary, complex matrices with $\det V = 1$)

2. Fields $\Psi = (\psi_1, \dots, \psi_n)^T$ are n-plets of Dirac fields ψ_i:

   $\Psi : \mathbb{R}^{1,3} \to \mathbb{C}^4 \otimes \mathbb{C}^n \simeq \mathbb{C}^{4n}$ and transforms as

$$\boxed{\tilde{\Psi}(x) = V(x)\,\Psi(x) = V_{ij}(x)\,\underbrace{\psi_j}_{\text{bispinor}}(x)}$$

   with $V : \mathbb{R}^{1,3} \to G$ arbitrary

3. G Lie group $\to$ Lie algebra $\mathfrak{g}$ with N Hermitian generators t^a ($n\times n$ matrixes, $a = 1, \dots, N$)

   that obey

$$\boxed{[t^a, t^b] = i f^{abc} t^c}$$

   Einstein notation

   with structure constants $f^{abc} \in \mathbb{C}$.

$\Rightarrow$ P.&S. pp. 495–502 Details on Lie-Groups and Lie-Algebras

$\rightarrow \quad V(x) = \exp[i\alpha^a(x)\, t^a] = 1 + i\alpha^a(x)\, t^a + \mathcal{O}(\alpha^2)$

4. The "comparator" is now a $n\times n$ unitary matrix with transformation

$$\tilde{U}(y,x) = V(y)\, U(y,x)\, V^\dagger(x) \qquad \text{and} \qquad U(y,y) = \mathbb{1} \qquad (9.4)$$

$\rightarrow$

$$U(x+\varepsilon n, x) = \mathbb{1} + ig\,\varepsilon n^\mu A_\mu^a t^a + \mathcal{O}(\varepsilon^2) \qquad (9.5)$$

g: arbitrary constant

A_μ^a: N vector fields (= gauge connections)

5. (9.2) $\rightarrow$ Covariant derivative:

$$\boxed{D_\mu \hat{=} \partial_\mu - ig A_\mu^a t^a} \qquad (9.6)$$

Often $A_\mu := A_\mu^a t^a$ ($n\times n$ matrix)

6. Transformation of A_μ^a:

a) (9.5) in (9.4)

b) Use $V(x+\varepsilon n)\,V^\dagger(x) \overset{\circ}{=} 1 + \varepsilon n^\mu V(x)\left[-\partial_\mu V^\dagger(x)\right] + \mathcal{O}(\varepsilon^2)$

$$\rightarrow \boxed{\tilde{A}_\mu = \tilde{A}_\mu^a t^a = V(x)\left[A_\mu^a t^a + \frac{i}{g}\partial_\mu\right] V^\dagger(x)} \quad \text{(exact)} \qquad (9.7)$$

acts on this

c) $\partial_\mu V^\dagger(x)$ not easy to evaluate $\rightarrow$ infinitesimal transformation $V^\dagger(x) \approx \mathbb{1}$:

$$V(x) = 1 + i\alpha^a(x)\, t^a + \mathcal{O}(\alpha^2)$$

$$\partial_\mu V^\dagger(x) = -i\,\partial_\mu \alpha^a(x)\, t^a + \mathcal{O}(\alpha^2)$$

with $f^{cba} = -f^{abc}$ $\rightarrow$

$$\boxed{\tilde{A}_\mu^a \overset{\circ}{\approx} A_\mu^a + \frac{1}{g}\partial_\mu \alpha^a + \underbrace{f^{abc} A_\mu^b \alpha^c}_{\text{New!}}} \quad \text{(infinitesimal)}$$

7. (9.7) in (9.6) $\rightarrow$ Transformation of $D_\mu \psi$:

$$\boxed{\tilde{D}_\mu \tilde{\psi} \overset{\circ}{=} V D_\mu \psi} \qquad (9.8)$$

$\rightarrow$ $D_\mu \psi$ transforms like ψ

Example: $\bar{\psi} D_\mu \psi$ is gauge invariant, $\bar{\psi} \not{D} \psi$ is both gauge and Lorentz invariant

8. Kinetic energy term for A^a_μ?

a) Iteration of (9.8) implies $\tilde{D}_\mu \tilde{D}_\nu \tilde{\psi} = V D_\mu D_\nu \psi$

$$\Rightarrow [\tilde{D}_\mu, \tilde{D}_\nu] \tilde{\psi} = V [D_\mu, D_\nu] \underset{V^+ V}{\wedge} \psi = V [D_\mu, D_\nu] V^+ \tilde{\psi}$$

$$\Rightarrow [\tilde{D}_\mu, \tilde{D}_\nu] = V [D_\mu, D_\nu] V^+ \qquad (9.9)$$

b) On the other hand: (9.6) $\rightarrow$

$$\boxed{-ig F^a_{\mu\nu} t^a := [D_\mu, D_\nu] \qquad \text{with } F^a_{\mu\nu} \overset{\circ}{=} \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu}$$

$F^a_{\mu\nu}$: N field-strength tensors (Note: $F_{\mu\nu} := F^a_{\mu\nu} t^a$ is a $n \times n$ matrix)

c) (9.9) $\rightarrow$ $$\boxed{\tilde{F}_{\mu\nu} = \tilde{F}^a_{\mu\nu} t^a = V F_{\mu\nu} V^\dagger}$$

$\rightarrow$ $F_{\mu\nu}$ is no longer gauge invariant

d) Simplest invariant term:

$$\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{Tr}\left[F^2\right] \equiv -\frac{1}{2} \mathrm{Tr}\left[\left(F^a_{\mu\nu} t^a\right)\left(F^{\mu\nu\, b} t^b\right) \right]$$

$\mathrm{Tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab}$ (P&S p. 498 ff.)

$$= -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2$$ (Yang-Mills theory)

Note 9.1

Important: $F^2 \sim (\partial A)^2 + \underbrace{f(\partial A)AA + f^2 AAAA}_{\text{Interactions}}$

→ Interacting QFT for $f \neq 0$ (= non-abelian)!

→ Gauge bosons scatter off each other

Example:

Quantum Chromodynamics [$G = SU(3)$] (⇒ last lecture)

Gauge bosons = Gluons → Pure gluon vertices:

→ Bound states of gluons
Glueballs

→ Related to "Yang-Mills Existence and Mass Gap" Problem at the Clay Mathematics Institute (Millennium Prize Problems)

9. Couple Dirac fermions to Yang-Mills gauge field:

$$\boxed{\mathcal{L}_{YM+D} = \bar{\Psi}(i\not{D} - m)\,\Psi - \frac{1}{4}\left(F^a_{\mu\nu}\right)^2}$$

Two parameters:

m: Fermion mass

g: Coupling constant (hidden D and F^2)

→ Yang-Mills theories describe all fundamental forces of the standard model!

(⊜ last lecture)

Note: $\not{D} = \gamma^\mu D_\mu = \partial_\mu \gamma^\mu \mathbb{1}_n - i g A^a_\mu \gamma^\mu t^a$

where $\gamma^\mu t^a \equiv \gamma^\mu \otimes t^a = \gamma^\mu_{\alpha\beta} \cdot t^a_{mn} = (\gamma^\mu t^a)_{(\alpha,m)(\beta,n)}$

and $\Psi : \mathbb{R}^{1,3} \to \mathbb{C}^4 \otimes \mathbb{C}^n$

Lorentz group: $\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$

Gauge group: $V = \exp\left(i\alpha^a t^a\right)$

$\tilde{\bar{\Psi}} = (V\Psi)^\dagger \gamma^0 = \Psi^\dagger V^\dagger \otimes \gamma^0$

$= \underbrace{\Psi^\dagger \gamma^0}_{\gamma^0\otimes\mathbb{1}_n} \underbrace{V^\dagger}_{\mathbb{1}_4\otimes V^\dagger} = \bar{\Psi} V^\dagger$

→ $\bar{\Psi}\Psi$ is gauge invariant

Note 9.2

The mass term A^2 is <u>not</u> allowed as it is not gauge invariant!

Recall $\frac{m}{2}\phi^2$ and $m\bar{\Psi}\Psi$ create mass.

→ Gauge bosons of Yang-Mills theories are massless

In QED fine: photon is massless

Problem:

The weak interaction is short-ranged, i.e., its gauge bosons $W^\pm$ and Z have mass!

Solution:

Higgs mechanism (⇒ next lecture)