10. Excursions

10.1 The Higgs Mechanism

10.1.1. Abelian Example: The Standard Approach

Goal: $\exists$ Abelian gauge theory to understand the Higgs mechanism

1. $\exists$ Maxwel theory coupled to a complex scalar field:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi)$$

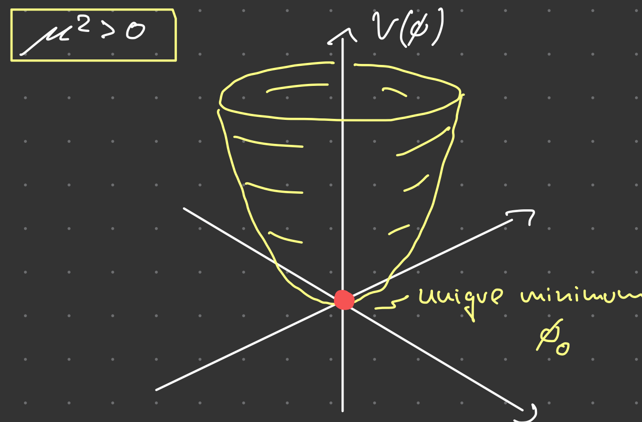
with potential $\quad V(\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4 \qquad (10.1)$

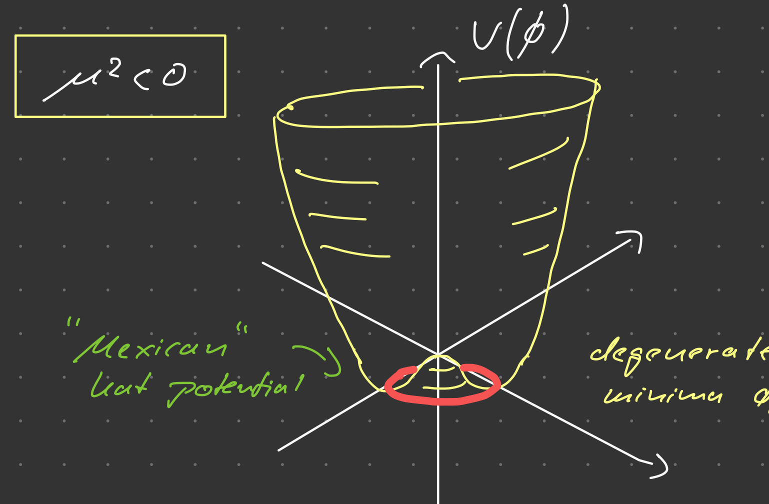
and $\quad D_\mu = \partial_\mu + ie A_\mu$

2. $\mathcal{L}$ is invariant under $U(1)$ gauge transformation

$$\tilde{\phi}(x) = e^{i\alpha(x)}\phi(x) \qquad \text{and} \qquad \tilde{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

3. § $V(\phi)$ in the complex plane $\phi \in \mathbb{C}$:





- $\mu^2 > 0$: Unique minimum with $\langle \phi \rangle = 0$

- $\mu^2 < 0$: $\longrightarrow$ Mexican hat potential:
  Degenerate minima with non-zero vacuum expectation value (VEV)

$$\phi_0 := \langle \phi \rangle \quad \text{and} \quad v := |\phi_0| = \sqrt{\frac{-\mu^2}{2\lambda}} \neq 0$$

$\longrightarrow$ Ground states are not symmetric under global phase rotations

$\longrightarrow$ Spontaneous symmetry breaking (SSB) of the <u>global</u> $U(1)$ symmetry

4. Aside: The Goldstone Theorem

If a global, continuous symmetry is spontaneously broken, there is one massless scalar (= Spin-0) particle for each broken symmetry generator; these particles are known as (Nambu-) Goldstone bosons.

"Proof by picture:"

Examples:

- Breaking of translational and rotational invariance in crystals $\rightarrow$ Transversal and longitudinal phonons
- Breaking of rotation symmetry in a ferromagnet

  $\rightarrow$ Magnons (= Spin waves)

Exception:

In conventional superconductors the global $U(1)$ symmetry is broken spontaneously ($\Leftrightarrow$ Ginzburg-Landau theory) – but there is no massless Goldstone boson!

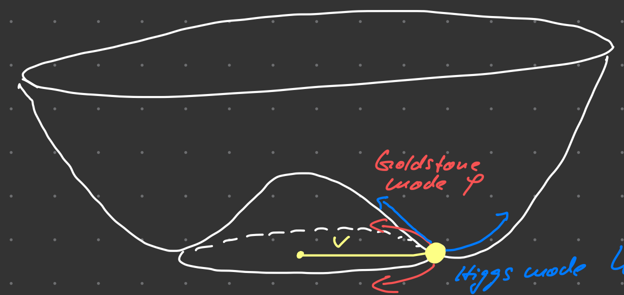
→ How can the Goldstone theorem fail?

→ Answer: Gauge symmetry & Higgs mechanism!

5. Assume that $\langle\phi\rangle = \phi_0 = v$ breaks the global $U(1)$ symmetry

→ Expand ϕ in small fluctuations around $\langle\phi\rangle$:

$$\phi(x) = [v + h(x)] \cdot e^{i\varphi(x)}$$



with two real fields:

- $h(x)$: Higgs field
- $\varphi(x)$: Goldstone boson

$$\rightarrow \quad \mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \left[(\partial_\mu + ieA_\mu)(v+h)e^{i\varphi}\right]\left[(\partial^\mu - ieA^\mu)(v+h)e^{-i\varphi}\right]$$

$$-\mu^2(v+h)^2 - \lambda(v+h)^4$$

$$\hat{=} \underbrace{-\frac{1}{4}(F_{\mu\nu})^2 + e^2v^2A_\mu^2}_{\text{Massive gauge field!}} + \underbrace{(\partial_\mu h)^2 - m_h^2 h^2}_{\text{Higgs field with mass } m_h^2 = 4\lambda v^2}$$

$$+ \underbrace{v^2(\partial_\mu\varphi)^2}_{\text{Massless Goldstone mode}} + \underbrace{2ev^2(\partial_\mu\varphi)A^\mu}_{\text{Quadratic coupling}} + \underbrace{\cdots\cdots}_{\text{Interactions}}$$

Note: Still gauge invariant under gauge transformation

$$\tilde{\varphi} = \varphi + \alpha \quad \text{and} \quad \tilde{A}_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha \quad \text{and} \quad \tilde{h} = h$$

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6. Fix the gauge in the unitary gauge $\phi \doteq \phi^* \; (\Leftrightarrow \varphi \doteq 0)$ with the gauge transformation $\alpha(x) = -\varphi(x)$

$$\tilde{\phi}(x) = e^{-i\varphi}\phi \quad \text{and} \quad \tilde{A}_\mu = A_\mu + \frac{1}{e}\partial_\mu\varphi(x)$$

$$\rightarrow \quad \boxed{\tilde{\mathcal{L}} = \underbrace{-\frac{1}{4}(F_{\mu\nu})^2 + e^2 v^2 A_\mu^2}_{\text{massive gauge field}} + \underbrace{(\partial_\mu h)^2 - m_h^2 h^2}_{\text{massive Higgs field}} + \text{Interactions}} \tag{10.2}$$

$\rightarrow$ Goldstone mode φ has disappeared!

<u>Reason:</u> φ is a <u>pure gauge</u> d.o.f. and therefore not physical!

<u>7. Consistency check:</u> Counting physical degrees of freedom:

$$\boxed{\begin{aligned} \#\,\text{d.o.f. before SSB} &= 2\ (\text{massless vector boson}) + 2\ (\text{complex scalar field}) = 4 \\ \#\,\text{d.o.f. after SSB} &= 3\ (\text{massive vector boson}) + 1\ (\text{real scalar Higgs field}) = 4 \end{aligned}}$$

Note 10.1

- We have seen that the Goldstone theorem is not valid for gauge theories
- The Higgs mechanism also describes conventional superconductivity as spontaneous $U(1)$ symmetry breaking in a charged superfluid ($\ni$ Ginzburg-Landau theory).

10.1.2. Bonus: A Gauge-Invariant Approach

1. $\mathcal{L}$ Again (10.1)

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu \phi|^2 - \mu^2 |\phi|^2 - \lambda |\phi|^4$$

2. Let $\mu^2 < 0 \to$ Classical ground state:

$$\phi_0(x) = e^{i\alpha(x)} \phi_0 \qquad \text{with} \quad \alpha(x) \text{ arbitrary} \quad (\alpha(x) \equiv 0)$$

and $|\phi_0| = \sqrt{\frac{-\mu^2}{2\lambda}} = v \neq 0$ (here $\phi_0 = v$)

3. $\cancel{x}$ Small fluctuations around ϕ_0 and introduce the new <u>real</u> fields $h(x)$, $\varphi(x)$ and $B_\mu(x)$:

$$\boxed{\phi(x) \equiv [v + h(x)] \cdot e^{i\varphi(x)} \quad \text{and} \quad B_\mu(x) \equiv A_\mu(x) + \frac{1}{e}\partial_\mu \varphi(x)}$$

→ Gauge transformations:

$\tilde{\varphi} = \varphi + \alpha \quad \longrightarrow$ pure gauge

$\tilde{h} = h \quad \longrightarrow$ gauge invariant

$\tilde{B}_\mu = B_\mu \quad \longrightarrow$ gauge invariant

Compare this to

$\tilde{\phi} = e^{i\alpha}\phi \quad \longrightarrow$ gauge dependent

$\tilde{A}_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha \longrightarrow$ gauge dependent

4. Express Lagrangian in new fields:

$B_\mu \overset{!}{=} A_\mu$

$$\boxed{\mathcal{L} \overset{\circ}{=} -\frac{1}{4}(B_{\mu\nu})^2 + e^2 v^2 B_\mu^2 + (\partial_\mu h)^2 - m_h^2 h^2 + \text{Interactions} = (10.2)}$$

$$B_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu \overset{\circ\circ}{=} \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$

$\longrightarrow$

- Gauge d.o.f. φ drops out (unconstrained by the Lagrangian)
- $\mathcal{L}$ is manifestly gauge-invariant
- B_μ is a massive vector boson
- h is a massive Higgs mode

5. <u>Take-Home-Message</u>:

> There is <u>no</u> spontaneous breaking of local gauge symmetries in the Higgs mechanism!

Note 10.2

The Higgs mechanism can be straightforwardly generalized to non-abelian gauge symmetries.