10.2 The Standard Model

Preliminaries:

- Define the chiral projectors

$$P_R := \frac{1}{2}(\mathbb{1}_4 + \gamma^5) \overset{\text{Weyl}}{=} \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \quad \text{and} \quad P_L := \frac{1}{2}(\mathbb{1}_4 - \gamma^5) \overset{\text{Weyl}}{=} \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix}$$

and the chiral fermion fields

$$\psi_R := P_R \psi \quad \text{and} \quad \psi_L := P_L \psi$$

- With $\bar\psi P_R = \bar\psi_L$ and $\bar\psi P_L = \bar\psi_R$ show that

$$\bar\psi (i\partial\!\!\!/ - m)\psi \overset{\circ}{=} \bar\psi_R (i\partial\!\!\!/)\psi_R + \bar\psi_L (i\partial\!\!\!/)\psi_L - m\,\bar\psi_L\psi_R - m\,\bar\psi_R\psi_L \qquad (10.3)$$

$\mathbb{1} = P_R + P_L$

- Note $[P_{R/L}, \Lambda_{\frac{1}{2}}] \overset{\circ}{=} 0$

→ Under additional (gauge) symmetries, the left- and right-handed fields $\psi_{L,R}$ (their multiplets) can transform under <u>different</u> representations of these new symmetry groups.

10.2.1. Overview

1. Field content :

- Fermions (= Spin-$\frac{1}{2}$):

Generation	I	II	III
Leptons	e_L e_R ν_{eL} (ν_{eR})	μ_L μ_R $\nu_{\mu L}$ $(\nu_{\mu R})$	τ_L τ_R $\nu_{\tau L}$ $(\nu_{\tau R})$
Quarks	u_L u_R d_L d_R	c_L c_R s_L s_R	t_L t_R b_L b_R

- **Vector bosons** (= Spin-1):

Force	Electroweak	Strong
Gauge group	$SU(2)_L \times U(1)_Y$	$SU(3)_C$
# Generators	$3+1=4$	8
Gauge fields	$\underbrace{W^i_\mu \ (i=1,2,3),\ B_\mu}_{\text{Before Higgs SSB}}$	$G^a_\mu \ (a=1,\dots,8)$
Gauge bosons	$\underbrace{\gamma,\ W^+, W^-,\ Z}_{\text{After Higgs SSB}}$	$8 \times$ Gluons

Warning: The gauge field B_μ of the $U(1)_Y$ symmetry does <u>not</u> correspond to the photon of QED!

- Scalar boson (= Spin-0):

$$\boxed{2 \times \underline{\text{Complex}} \text{ Higgs fields } \phi^+, \phi^0 \xrightarrow{3\times\text{ SSB}} 1 \times \underline{\text{Real}} \text{ Higgs boson } h}$$

2. Question: How to put this "chaos" into a consistent (= relativistic, renormalizable) QFT?

3. Answer:

$$\boxed{\mathcal{L}_{SM} = \mathcal{L}_{EWS} + \mathcal{L}_{QCD} \qquad \text{Standard model}}$$

4. Two parts:

   - Electroweak Standard Model $\mathcal{L}_{EWS}$ = Glashow-Weinberg-Salam (GWS) Theory = Unification of weak & electromagnetic force
   - Quantum Chromodynamics $\mathcal{L}_{QCD}$ = strong force

10.2.2. The Glashow-Weinberg-Salam Theory

Goal: Generalize the Higgs mechanism to the Standard model

1. Lagrangian

$$\boxed{\mathcal{L}_{EWS} = \mathcal{L}_{Fermion} + \mathcal{L}_{Yang\text{-}Mills} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa}}$$

2. Gauge symmetry

$$\boxed{\underbrace{SU(2)_L}_{\text{Weak isospin}} \times \underbrace{U(1)_Y}_{\text{Weak Hypercharge}}}$$

- $SU(2)_L \longrightarrow$ 3 generators T^i, $i = 1,2,3$ with $[T^i, T^j] = i\varepsilon^{ijk}T^k$

→ Irreducible representations:

* 1D : Trivial representation $\hat{T}^i = 0$ (= Singlet rep.)
* 2D : Pauli matrices $\hat{T}^i = \frac{\sigma^i}{2}$ (= Doublet rep.)

→ Eigenvalues of $\hat{T}^3$ = The weak isospin T^3 ($T^3 = \pm\frac{1}{2}$ for doublet and $T^3 = 0$ for singlet)

- $U(1)_Y \to 1$ generator Y

$$[Y, T^i] = 0 \quad \text{for all } i$$

Schur's lemma

$$\to \hat{Y} = \text{Number} \times \mathbb{1} = \text{Hypercharge } Y \times \mathbb{1}$$

3. $SU(2)_L$ Representations:

- Left-handed fields = Isospin doublets:

$$\Psi_L = \underbrace{\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}}_{\text{Gen I}}, \underbrace{\begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}}_{\text{Gen II}}, \underbrace{\begin{pmatrix} t_L \\ b_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}}_{\text{Gen III}} \tag{10.4}$$

$\to$ Weak isospin: $T^3(\nu_{eL}) = +\frac{1}{2}$ and $T^3(e_L) = -\frac{1}{2}$ . . .

Note: $\not\exists \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$

$$\nu_{eL}(x) = \psi_L(x) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \underbrace{L^2(\mathbb{R}^{1,3})}_{\text{Field}} \otimes \underbrace{\mathbb{C}^4}_{\text{Dirac bispinor}} \otimes \underbrace{\mathbb{C}^2_L}_{SU(2)_L \text{ doublet rep.}}$$

$$T^3(\nu_{eL}) = +\frac{1}{2} :\Longleftrightarrow \hat{T}^3 \nu_{eL}(x) = \psi_L(x) \otimes \hat{T}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{1}{2} \nu_{eL}(x)$$

$$\nu_{eL} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (\text{flavours})$$

- Right-handed fields = Isospin singlets:

$$\psi_R = \underbrace{u_R, d_R, e_R}_{\text{Gen I}}, \underbrace{c_R, s_R, \mu_R}_{\text{Gen II}}, \underbrace{t_R, b_R, \tau_R}_{\text{Gen III}} \qquad (10.5)$$

→ Weak isospin: $T^3(e_R) = 0$ .....

- Higgs fields = Isospin doublet:

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

→ Weak isospin: $T^3(\phi^+) = +\frac{1}{2}$ and $T^3(\phi^0) = -\frac{1}{2}$

→ Gauge transformations on fields:

Left-handed doublet: $\tilde{\Psi}_L = e^{i\hat{Y}_L \alpha(x)} \underbrace{e^{i\hat{T}^i \beta^i(x)}}_{= V_L(x)} \Psi_L$

Right-handed singlet: $\tilde{\psi}_R = e^{i\hat{Y}_R \alpha(x)} \psi_R$

Higgs doublet: $\tilde{\Phi} = e^{i\hat{Y}_H \alpha(x)} e^{i\hat{T}^i \beta^i(x)} \Phi$

where

$\hat{Y}_L = Y \cdot \mathbb{1}_{2\times 2}$

$\hat{Y}_R = Y \cdot 1$

$\hat{Y}_H = Y \cdot \mathbb{1}_{2\times 2}$

Note:

The weak hypercharge Y is a fixed number for each irrep, e.g., $Y(u_L) = Y(d_L)$, but can differ for different irreps: $Y(u_L) \neq Y(e_L)$

4. Kinetic energy for fermions & Minimal coupling:

$$\boxed{\mathcal{L}_{Fermion} = \sum_{\Psi_L} \bar{\Psi}_L (i \not{D}_L) \Psi_L + \sum_{\psi_R} \bar{\psi}_R (i \not{D}_R) \psi_R} \tag{10.6}$$

with covariant derivatives

$$D_{L\mu} = \partial_\mu - ig W_\mu^i \hat{T}^i - ig' B_\mu \hat{Y}_L$$

$$D_{R\mu} = \partial_\mu \qquad\qquad\qquad - ig' B_\mu \hat{Y}_R$$

g / W_μ^i : coupling constant / gauge field for weak isospin

g' / B_μ : — " — / — " — for weak hypercharge

→ Transformation of the gauge fields:

$$W_\mu \equiv W_\mu^i \hat{T}^i$$

$$\boxed{\tilde{B}_\mu = B_\mu + \frac{1}{g'} \partial_\mu \alpha \quad \text{and} \quad \tilde{W}_\mu = V_L \left[W_\mu + \frac{i}{g} \partial_\mu \right] V_L^\dagger}$$

5. Dirac mass terms? Should be of the form

$$m \left(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L \right) \rightarrow \text{Undefined!}$$

∄ Elementary terms of the form $\bar{X}_L Y_R$ with X, Y Dirac spinors

→ Not $SU(2)_L$ gauge invariant since

* X_L is component of a $SU(2)$ doublet
* Y_R transforms as a $SU(2)$ singlet

→ $\bar{X}_L X_R$ is not a $SU(2)$ singlet

→ We cannot add Dirac mass terms to the Lagrangian!

→ Need Yukawa couplings & Higgs mechanism (⇒ below)

Note:

$E p^2$ not Lorentz scalar because $E = p^0$ is component of a four vector

6. Kinetic energy for gauge bosons:

$$\boxed{\begin{aligned} \mathcal{L}_{\text{Yang-Mills}} &= -\frac{1}{4}(F^{\mu\nu})^2 - \frac{1}{4}(G^i_{\mu\nu})^2 \\ \text{with} \quad F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ G^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + \underbrace{g\,\varepsilon^{ijk} W^j_\mu W^k_\nu}_{\text{interactions between gauge bosons}} \end{aligned}}$$

7. Higgs field:

$$\boxed{\mathcal{L}_{\text{Higgs}} = (D^\mu_H \Phi)^\dagger (D_{H\mu}\Phi) - \mu^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger\Phi)^2} \qquad (10.7)$$

with covariant derivative

$$D_{H\mu} = \partial_\mu - i g W^i_\mu T^i - i g' B_\mu \hat{Y}_H$$

8. Higgs mechanism Part I: Masses for the gauge bosons

a) Let $\mu^2 < 0$ $\longrightarrow$ Non-zero VEV of Higgs field

W.l.o.g. $\quad \langle \Phi \rangle = \Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad$ with $\quad v = \sqrt{\frac{-\mu^2}{\lambda}}$

b) Define the electric charge (operator)

$$\boxed{Q = T^3 + Y \quad \in su(2)_L \oplus u(1)_Y} \qquad (10.8)$$

$\rightarrow$ Choose $Y(\Phi) = +\frac{1}{2}$ so that

$$\underbrace{\hat{Q}}_{\hat{T}^3 + \underbrace{\hat{Y}_H}_{\frac{1}{2} \cdot \mathbb{1}_{2\times 2}}} \Phi_0 = \left(-\frac{1}{2} + \frac{1}{2}\right) \Phi_0 = 0 \quad \Longrightarrow \quad e^{i \hat{Q} \alpha(x)} \Phi_0 = \Phi_0$$

Note: $\quad Q(\phi^0) = 0 \quad$ but $\quad Q(\phi^+) = +1$

→ Gauge symmetry $U(1)_Q$ generated by Q is unbroken:

$$\boxed{SU(2)_L \times U(1)_Y \xrightarrow{3\times \text{SSB}} U(1)_Q}$$

Unbroken gauge group of QED

Conclusion: The generator of $U(1)_Y$ (weak hypercharge Y) and the generator of $U(1)_Q$ (electric charge) are not the same!

c) ∫ Fluctuations of Φ around Φ_0 in the unitary gauge:

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

$h(x)$: real scalar Higgs field (Excitations of $h(x)$: The Higgs boson)

d) $\Phi(x)$ in (10.7)

$$(D_H^\mu \Phi)^\dagger (D_{H\mu} \Phi) \stackrel{\circ}{=} \frac{v^2}{8}\left\{ g^2\left[(W_\mu^1)^2 + (W_\mu^2)^2\right] + \left(-g W_\mu^3 + g' B_\mu\right)^2 \right\} + \dots \qquad (10.9)$$

e) Define the new field

$$\boxed{\begin{aligned} W_\mu^\pm &:= \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i W_\mu^2\right) \\ Z_\mu &:= \frac{1}{\sqrt{g^2+g'^2}}\left(g W_\mu^3 - g' B_\mu\right) \\ A_\mu &:= \frac{1}{\sqrt{g^2+g'^2}}\left(g' W_\mu^3 + g B_\mu\right) \end{aligned}}$$

Note:

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$$

= weak mixing angle

= Weinberg angle

$$\rightarrow (10.9) \stackrel{\circ}{=} \underbrace{\left(\frac{gv}{2}\right)^2}_{m_W^2} W_\mu^+ W^{-\mu} + \frac{1}{2}\underbrace{\left(\frac{v}{2}\right)^2 (g^2+g'^2)}_{m_Z^2} (Z_\mu)^2 + \dots$$

and

$$\boxed{D_{H\mu} = \partial_\mu - (\dots) - i \underbrace{\frac{g g'}{\sqrt{g^2+g'^2}}}_{\text{Electron charge } e} A_\mu \hat{Q}}$$

- A_μ: massless, neutral ($Q=0$) gauge field of QED
- $W^\pm_\mu$: massive, charged ($Q=\pm 1$) gauge boson of weak interaction
- Z_μ: massive, neutral ($Q=0$) gauge boson of weak interaction

9. <u>Interlude:</u> With (10.8) we can fix the hypercharge Y by the (observed) electric charge Q.

Examples:

$$Y(e_L) = Q(e_L) - T^3(e_L) = -1 - \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

$$Y(e_R) = Q(e_R) - T^3(e_R) = -1 - 0 = -1$$

10. <u>Higgs mechanism Part II:</u> Masses for the fermions

a) How to form a gauge invariant ($= SU(2)_L$ singlet $+ Y=0$) term including left- and right-handed fermions?

→ Couple left-handed fermion doublet, Higgs doublet, and right-handed fermion singlet via a Yukawa term:

$$-\gamma_e\,(\overline{\Psi_L}\cdot\Phi)\cdot e_R + h.c. \qquad (10.10)$$

$$\Psi_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$$

γ_e: coupling constant

- $Y(\Phi) + Y(e_R) - Y(\Psi_L) = \frac{1}{2} - 1 - \left(-\frac{1}{2}\right) = 0$

- $(\overline{\Psi_L}\cdot\Phi)\,e_R = \underbrace{(\overline{\nu_{eL}}\ \overline{e_L})\cdot\begin{pmatrix}\Phi^+ \\ \Phi_0\end{pmatrix}}_{SU(2)_L \text{ singlet}}\cdot e_R = \underbrace{\overbrace{\phi^+\cdot\overline{\nu_{eL}}e_R + \phi^0\cdot\overline{e_L}e_R}^{\text{Yukawa interaction (cf. } A_\mu\,\overline{\psi}\gamma^\mu\psi)}}_{\text{Scalars} \times \text{Dirac inner products}}$

Higgs mechanism: $(\phi^+ \mapsto 0,\ \phi^0 \mapsto \frac{1}{\sqrt{2}}v)$

$$(10.10) = -\frac{\gamma_e v}{\sqrt{2}}\left(\overline{e_L}e_R + \overline{e_R}e_L\right) + \ldots\ldots$$

with fermion mass $m_e = \dfrac{\gamma_e v}{\sqrt{2}}$

b) In general, we can couple different fermion generations:

$$\boxed{\mathcal{L}_{\text{Yukawa}} = -\Gamma^{u}_{mn}\,\overline{Q}_L^{m}\,\hat{\Phi}\,u_R^{n} - \Gamma^{d}_{mn}\,\overline{Q}_L^{m}\,\Phi\,d_R^{n} - \Gamma^{l}_{mn}\,\overline{L}_L^{m}\,\Phi\,l_R^{n}} \qquad (10.11)$$

- $m,n \in \{I, II, III\}$: fermion generations
- $x \in \{u, d, l, \nu\}$: fermion types

  Example: $l_R^{I} = e_R$, $l_R^{II} = \mu_R$, $u_R^{I} = u_R$, $u_R^{II} = c_R$, ...

- Γ^{x}_{mn} : coupling constants

  Example: $\Gamma^{l}_{I,I} = \gamma_e$ from above

- Q_L^m, L_L^m: left-handed quark- resp. lepton doublets of generation m

  Example: $\overline{Q}_L^{I} = (\overline{u}_L\ \overline{d}_L)$ and $\overline{L}_L^{II} = (\nu_{\mu L}\ \mu_L)$ ...

- $\hat{\Phi}_i \equiv \varepsilon^{ij}\Phi_j^*$: Higgs doublet with opposite hypercharge: $Y(\hat{\Phi}) = -\frac{1}{2}$

c) The Yukawa coupling (10.11)....

- ... generate mass terms for quarks and charged leptons
- ... cannot generate mass terms for neutrinos if there are no right-handed neutrinos
- ... leads to generation-changing transitions of quarks (⊜ CKM mixing matrix)

10.2.3. Quantum Chromodynamics

1. Gauge symmetry :

$$\boxed{\underbrace{SU(3)_c}_{\text{Color charge}}}$$

→ 8 generators U^a, $a = 1, ..., 8$ with $[U^a, U^b] = i f^{abc} U^c$

→ Irreducible representations:

- 1D: Trivial representation $\hat{U}^a = 0$ (= Singlet representation)
- 3D: Defining representation $\hat{U}^a = \frac{\lambda_a}{2}$ with 3x3 Hermitian Gell-Mann matrices λ_a (= Triplet representation)

2. Field representation:

- Quarks $=$ $SU(3)_c$ triplets

$$q = \begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix} \quad \text{for} \quad q \in \{u, d, c, s, t, b\}$$

with colors r (red), g (green), b (blue)

- Leptons, Higgs $=$ $SU(3)_c$ singlets $\longrightarrow$ Ignore leptons for QCD

$\rightarrow$ Gauge transformation of fields:

$$\boxed{\text{Quark triplet:} \qquad \tilde{q} = \underbrace{e^{i\hat{\lambda}^a \beta^a(x)}}_{U_c(x)} q}$$

3. Lagrangian:

$$\boxed{\mathcal{L}_{QCD} = \sum_q \bar{q}\,(i \not{D}_c)\, q - \frac{1}{4}\left(G^a_{\mu\nu}\right)^2}$$

with covariant derivative

$$D_{c\,\mu} = \partial_\mu - i g_s\, G^a_\mu\, \hat{\lambda}^a$$

g_s: coupling constant of the strong force

G^a_μ: 8 gauge fields $\longrightarrow$ 8 gauge bosons = 8 Gluons

Gauge field strength:

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu$$

Note 10.3

- No additional Higgs mechanism:
  * Quark masses are generated by electroweak SSB
  * Gluons are massless
- Gluons carry color charges and can therefore interact with each other (⊜ Note 9.1)
- Renormalization: Let $\alpha_S = \frac{g_S^2}{4\pi}$, then

$$\alpha_S^{\text{eff}}(q^2) \xrightarrow{q^2 \to \infty} 0 \quad \to \text{Asymptotic freedom}$$

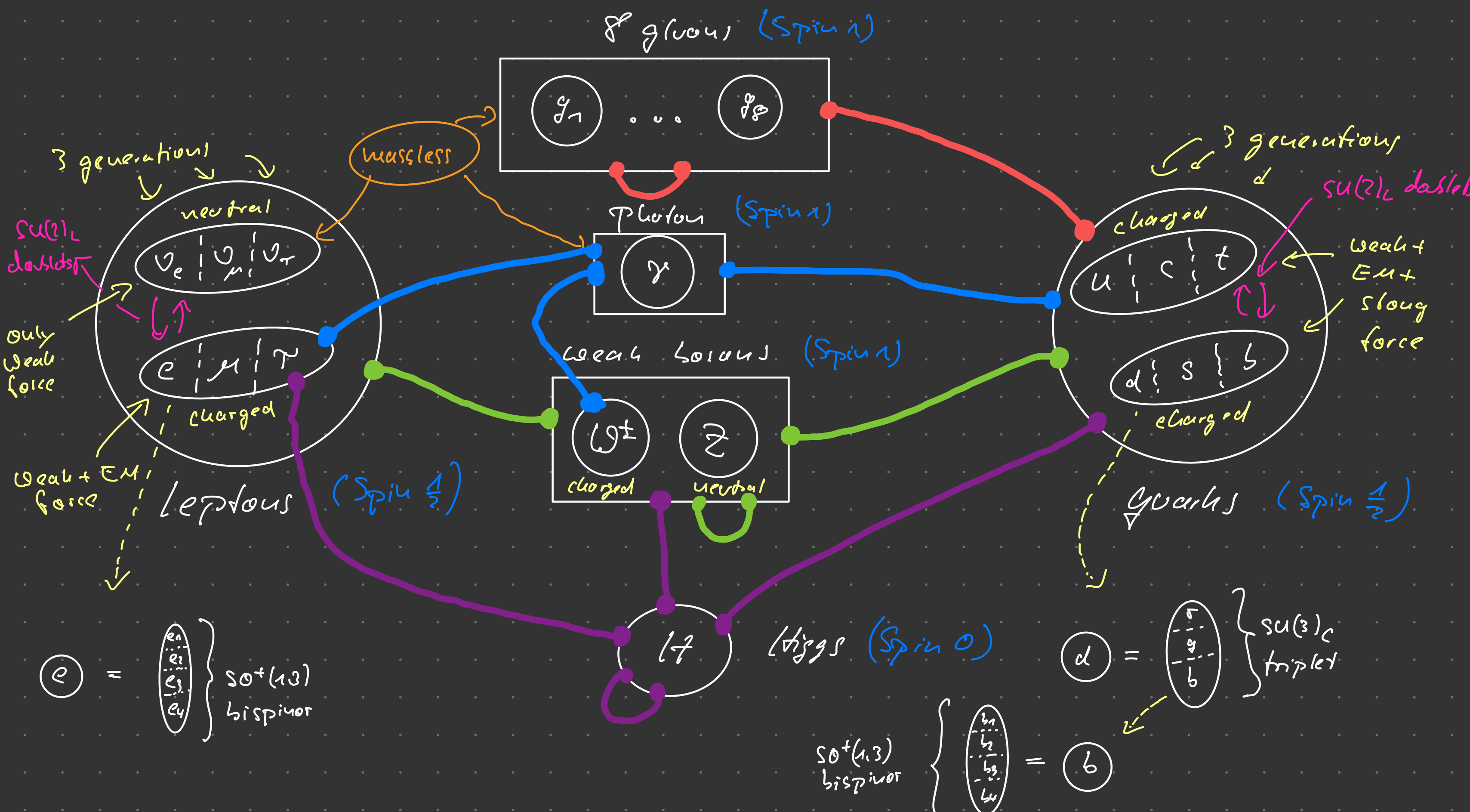
$$\alpha_S^{\text{eff}}(q^2) \xrightarrow{q^2 \to 0} \infty \quad \to \text{Confinement}$$

10.2.4 Summary

- Gauge symmetry group of the standard model:

$$\boxed{\begin{aligned} &\underbrace{\underbrace{SU(2)_L}_{\text{Weak isospin}} \times \underbrace{U(1)_Y}_{\text{Weak Hypercharge}}} \times \underbrace{SU(3)_C}_{\text{Color charge}} \\ &\xrightarrow{\text{Electroweak SSB}} \underbrace{U(1)_Q}_{\text{Electric charge}} \end{aligned}}$$

- Fermions and their interactions:



In total there are

$$[2 + 2 \times 3] \times 3 = 24 \text{ Dirac bispinors}$$

each consisting of 4 complex fields $\longrightarrow$ 96 complex fields for the fermions

- The standard model Lagrangian $\mathcal{L}_{SM}$ contains 18 parameters that cannot be derived but must be provided by experiments:

  * 9 $\times$ Fermion masses: $m_e, m_u, \ldots$

  * 1 $\times$ Higgs mass $m_h \approx 125\,\text{GeV}$

  * 1 $\times$ Higgs field VEV v

  * 3 $\times$ Gauge field couplings: g, g', g_s

  * 4 $\times$ CKM matrix parameters

$\rightarrow$ SM not good candidate for a fundamental theory

$\rightarrow$ GUT should allow for computation ab initio