2. The Klein-Gordon Field

2.1. Quantization

1. Theory:

   a) Real field $\phi(x)$

   b) Lagrangian: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$

   c) EOM: $(\partial^2 + m^2)\phi = 0$ (Klein-Gordon equation)

      $\uparrow \; \partial_\mu \partial^\mu$

   d) Hamiltonian: $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2$

2. Canonical quantization: $\left([x_i, p_j] = i \, \delta_{ij} \right)$

$$[\phi(\vec{x}), \pi(\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$[\phi(\vec{x}), \phi(\vec{y})] = 0 \qquad (2.1)$$

$$[\pi(\vec{x}), \pi(\vec{y})] = 0$$

Real field $\Leftrightarrow$ Hermitian operator

$$\phi^\dagger = \phi$$

$$\pi^\dagger = \phi$$

3. Goal: Spectrum of Hamiltonian?

4. Motivation:

a) Fourier transform of KG equation in space:

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \tilde{\phi}(\vec{p},t)$$

Then

$$\Big[\partial_t^2 + \underbrace{(\vec{p}^{\,2} + m^2)}_{\omega_p^2} \Big] \tilde{\phi}(\vec{p},t) = 0$$

$\longrightarrow$ decoupled harmonic oscillators with frequency

(and constraint $\tilde{\phi}^*(\vec{p},t) = \tilde{\phi}(-\vec{p},t)$ since $\phi^* = \phi$)

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}$$

b) $\measuredangle$ Hamiltonian $H_{SHO} = \frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}\omega^2\tilde{\phi}^2$ and introduce

$$\tilde{\phi} = \frac{1}{\sqrt{2\omega}}(a + a^+) \quad \text{and} \quad \tilde{\pi} = -i\sqrt{\frac{\omega}{2}}(a - a^+) \quad \text{with} \quad [a, a^+] = 1$$

$$\rightarrow H_{SHO} = \omega\left(a^+a + \frac{1}{2}\right)$$

5. This motivates the <u>Field operators</u>:

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_p e^{i\vec{p}\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\vec{x}} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \underbrace{\frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} + a_{-\vec{p}}^\dagger \right)}_{\tilde{\phi}(\vec{p})} e^{i\vec{p}\vec{x}} \qquad (2.2)$$

with <u>momentum modes</u>

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \underbrace{(-i)\sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} - a_{-\vec{p}}^\dagger \right)}_{\tilde{\pi}(\vec{p})} e^{i\vec{p}\vec{x}}$$

$$\boxed{[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})}$$

(2,3)

$$[a_{\vec{p}}, a_{\vec{q}}] = 0$$

$$[a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0$$

$\overset{\circ}{\longrightarrow}$ (2,3) $\wedge$ (2,2) $\Longrightarrow$ (2,1)

6. <u>Hamiltonian</u>:

$\propto \delta(0) = \infty$

$$\boxed{H \overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3} \, \omega_{\vec{p}} \left\{ a_{\vec{p}}^\dagger a_{\vec{p}} + \frac{1}{2} [a_{\vec{p}}, a_{\vec{p}}^\dagger] \right\}}$$

7. Eigenstates & Spectrum

a) $\overset{\circ}{\longrightarrow} [H, a^\dagger_{\vec p}] = \omega_{\vec p}\, a^\dagger_{\vec p}$

b) Vacuum $|0\rangle \longrightarrow$ Eigenstates $a^\dagger_{\vec p}\, a^\dagger_{\vec q} \ldots\ldots |0\rangle$

c) Energy: $E_{\vec p} = \omega_{\vec p} = \sqrt{|\vec p|^2 + m^2}$

d) (Kinetic) momentum:

$$P^i = \int d^3x\, \pi(x)(-\partial_i)\phi(x) \overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3}\, p^i\, a^\dagger_{\vec p} a_{\vec p} \qquad (2,4)$$

e) Statistics: $a^\dagger_{\vec p}\, a^\dagger_{\vec q}\, |0\rangle = a^\dagger_{\vec q}\, a^\dagger_{\vec p}\, |0\rangle$

$\longrightarrow$ Excitations $a^\dagger_{\vec p}$ commute and carry energy and momentum

$\longrightarrow$ Bosonic particles (in momentum space)

8. Normalization

$\sqrt{\vec{p}^2+m^2}$

a) $\forall\ \Lambda = R' L_3(\beta) R \in SO(1,3) \longrightarrow p' = (E_{\vec{p}'}, \vec{p}') = \Lambda p$ with $p = (E_{\vec{p}}, \vec{p})$

b) Jacobian in space: $\det\left(\frac{\partial \vec{p}'}{\partial \vec{p}}\right) \hat{=} \frac{d p_3'}{d p_3} \hat{=} \frac{E_{\vec{p}'}}{E_{\vec{p}}}$

$$\rightarrow \quad \delta^{(3)}(\vec{p}-\vec{q}) = \frac{E_{\vec{p}'}}{E_{\vec{p}}}\, \delta^{(3)}(\vec{p}'-\vec{q}')$$

$\rightarrow$ $\delta^{(3)}(\vec{p}-\vec{q})$ is not Lorentz invariant <u>but</u> $E_{\vec{p}}\, \delta^{(3)}(\vec{p}-\vec{q})$ is!

c) <u>Single-particle states:</u> $|\vec{p}\rangle := \sqrt{2E_{\vec{p}}}\, a^\dagger_{\vec{p}} |0\rangle \Rightarrow \langle \vec{p}|\vec{q}\rangle = (2\pi)^3 (2E_{\vec{p}})\, \delta^{(3)}(\vec{p}-\vec{q})$

9. Lorentz transformations $\Lambda \in SO(1,3)$:

$$U(\Lambda)|\vec{p}\rangle = |\Lambda\vec{p}\rangle \quad \Longleftrightarrow \quad \boxed{U(\Lambda)\, a^\dagger_{\vec{p}}\, U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}} \cdot a^\dagger_{\Lambda\vec{p}}}$$

$(\Lambda\vec{p})^i = \Lambda^i{}_j p^j$

10. Interpretation of $\phi(\vec{x})$:

$$\phi(\vec{x})\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\vec{p}}}\,e^{-i\vec{p}\vec{x}}\,|\vec{p}\rangle$$

For non-relativistic $|\vec{p}| \ll m \Rightarrow E_p \approx \text{const}$

$\rightarrow$ $\phi(\vec{x})$ creates a particle at position $\vec{x}$

("position-space representation" $\langle 0|\phi(\vec{x})|\vec{p}\rangle \overset{\circ}{=} e^{i\vec{p}\vec{x}}$

Note 2.1

1. Projector on single-particle sector: $\mathbb{1}_1 = \int \frac{d^3p}{(2\pi)^3}\,|\vec{p}\rangle\,\frac{1}{2E_{\vec{p}}}\,\langle\vec{p}|$
2. $\forall$ $f(p)$ Lorentz invariant $\longrightarrow$ $\int \frac{d^3p}{(2\pi)^3}\,\frac{f(p)}{2E_{\vec{p}}}$ is Lorentz invariant

2.2 The KG Field in Space-Time

1. Heisenberg operators: $\phi(x) = \phi(\vec{x},t) = e^{iHt}\phi(\vec{x})\,e^{-iHt}$

2. Heisenberg equation: $i\partial_t \mathcal{O} = [\mathcal{O}, H]$ for $\mathcal{O} = \phi, \pi$ yields

- $$i\partial_t \phi(x) = \left[\phi(x), \int d^3y \left\{ \tfrac{1}{2}\pi^2(\vec{y},t) + \tfrac{1}{2}(\nabla\phi(\vec{y},t))^2 + \tfrac{1}{2}m^2\phi^2(\vec{y},t)\right\}\right]$$
$$= \int d^3y \; i\delta^{(3)}(\vec{x}-\vec{y})\,\pi(\vec{y},t)$$
$$= i\pi(x)$$

- $$i\partial_t \pi(x) \overset{\circ}{=} -i(-\nabla^2 + m^2)\,\phi(x)$$

$$\boxed{\Rightarrow \quad (\partial_t^2 - \nabla^2 + m^2)\,\phi(x) = 0 \qquad \text{(Klein-Gordon equation)}}$$

3. Time evolution of modes:

$$e^{iHt}a_{\vec{p}}\,e^{-iHt} = a_{\vec{p}}\,e^{-iE_{\vec{p}}t}$$
$$e^{iHt}a^{\dagger}_{\vec{p}}\,e^{-iHt} = a^{\dagger}_{\vec{p}}\,e^{+iE_{\vec{p}}t}$$

4. Field operators:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \left(a_{\vec p}\, e^{-ipx} + a_{\vec p}^{\dagger}\, e^{ipx} \right) \Big|_{p^0 = E_{\vec p}}$$

$$\pi(x) = \partial_t \phi(x)$$

$$px = p^\mu x_\mu = E_{\vec p}\, t - \vec p \cdot \vec x$$

Note 2.2:

1. Hamiltonian generates time translations:

$$\phi(\vec x, t) = e^{iHt} \underbrace{\phi(\vec x, 0)}_{\phi(\vec x)} e^{-iHt}$$

2. Total momentum operator generates space translations:

$$\phi(\vec x) \hat{=} e^{-i\vec P \vec x} \phi(\vec 0)\, e^{i\vec P \vec x}$$

3. $\rightarrow$ Four-momentum operator generates space-time translations:

$$\phi(x) = e^{iPx} \phi(0)\, e^{-iPx}$$

$P^\mu = (H, \vec P)$ as defined in (2.4)

Note 2.3

- $e^{-ipx} \longleftrightarrow$ positive frequency solution of KG equation $\longleftrightarrow$ annihilation operator $a_{\vec{p}}$
- $e^{+ipx} \longleftrightarrow$ negative frequency solution of KG equation $\longleftrightarrow$ creation operator $a^{\dagger}_{\vec{p}}$