Causality

* Amplitude for a particle to propagate from y to x:

$$\boxed{D(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle \stackrel{\wedge}{=} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip(x-y)}} \qquad (2.5)$$

D is Lorentz invariant: $D(\Lambda(x-y)) = D(x-y)$ for $\Lambda \in SO^+(1,3)$

1. Time-like distance: $x^0-y^0=t$ and $\vec{x}-\vec{y}=0$

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}}\, e^{-i\underbrace{\sqrt{p^2+m^2}}_{E}t}$$

$$\overset{E=E(p)}{=} \frac{1}{4\pi^2}\int_m^\infty dE\, \sqrt{E^2-m^2}\, e^{-iEt}$$

$$\overset{t\to\infty}{\neq} 0 \qquad \longrightarrow \text{does not vanish}$$

$$\sim e^{-imt} \qquad \longrightarrow \text{propagation possible}$$

2. Space-like distance: $x^0 - y^0 = 0$ and $\vec{x} - \vec{y} = \vec{r}$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}\vec{r}} \longrightarrow p \cdot r \cdot \cos\theta$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \, \frac{p^2}{2E_p} \, \frac{e^{ipr} - e^{-ipr}}{ipr}$$

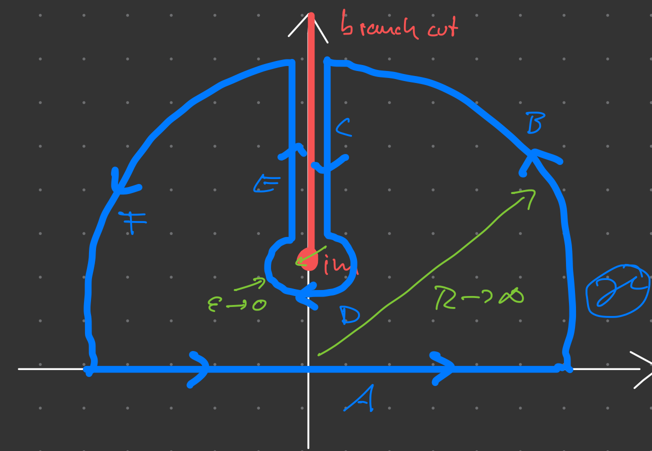
$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} dp \, \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}}$$

$$= -2C = \frac{-i}{(2\pi)^2 r} \int_{im}^{\infty} dp \, \frac{p \cdot e^{ipr}}{\sqrt{p^2+m^2}}$$

$\rho = -ip$

$$= \frac{1}{4\pi^2 r} \int_m^\infty d\rho \, \frac{\rho\, e^{-\rho r}}{\sqrt{\rho^2 - m^2}}$$

$$\overset{r\to\infty}{\sim} e^{-mr}$$



Cauchy's integral theorem:

$$\oint_\gamma f(z) = A + \underbrace{B}_{0} + C + \underbrace{D}_{0} + \underbrace{E}_{=C} + \underbrace{F}_{0} = 0$$

$$= A + 2C = 0$$

→ vanishes exponentially (but non-zero!) → Problem?

3. ☓ Measurements A and B: can affect each other iff $[A,B] \neq 0$

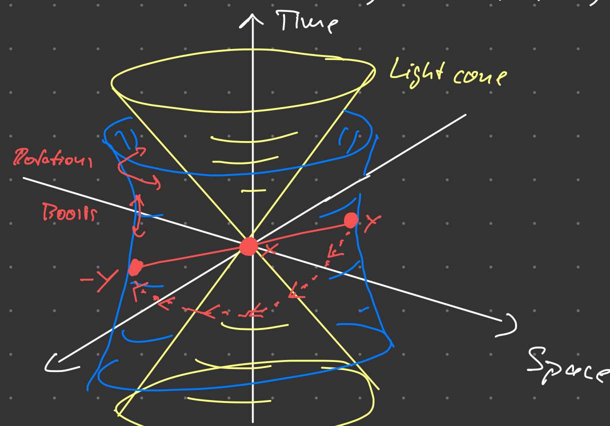
Simplest choice: $A = \phi(x)$, $B = \phi(y)$ $\quad (\pi = \partial_t \phi)$

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_{\vec q}} \left[a_{\vec p} e^{-ipx} + a^\dagger_{\vec p} e^{ipx}, \; a_{\vec q} e^{-iqy} + a^\dagger_{\vec q} e^{iqy} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip(x-y)} - e^{ip(x-y)} \right\}$$

$$= D(x-y) - D(y-x)$$

Let $(x-y)^2 < 0$ space-like $\rightarrow \exists \Lambda^* \in SO^+(1,3) : \Lambda^*(x-y) = -(x-y)$



Then

$$\boxed{\begin{aligned}[\phi(x),\phi(y)] &= D(x-y) - D(y-x) \\ &= D(x-y) - D(\Lambda^*(y-x)) \\ &\overset{(x-y)^2<0}{=} D(x-y) - D(x-y) = 0 \qquad (\to \text{Causality})\end{aligned}}$$

The Propagator

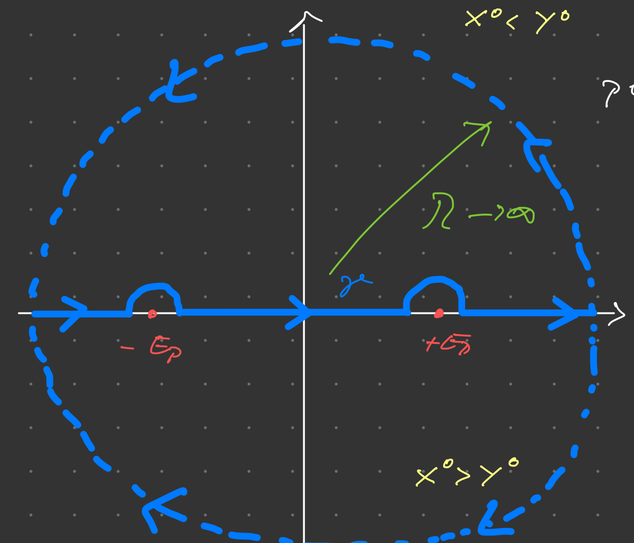
1. Since $[\phi(x),\phi(y)] \propto \mathbb{1}$ ($\to$ is a "c-number")

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}\left[e^{-ip(x-y)} - e^{ip(x-y)}\right]$$

$$\overset{\vec p \to -\vec p}{=} \int \frac{d^3p}{(2\pi)^3}\left\{ \frac{e^{-ip(x-y)}\big|_{p^0=E_{\vec p}}}{2E_{\vec p}} + \frac{e^{-ip(x-y)}\big|_{p^0=-E_{\vec p}}}{-2E_{\vec p}} \right\}$$

$$\overset{x^0>y^0}{=} \int \frac{d^3p}{(2\pi)^3}\int_\gamma \frac{d\overset{\in\mathbb{C}}{p^0}}{2\pi i}\,\frac{-1}{\underbrace{p^2-m^2}_{(p^0-E_{\vec p})(p^0+E_{\vec p}) = (p^0)^2-(E_{\vec p})^2}}\,e^{-ip(x-y)}$$

$$= \int_\gamma \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2}\,e^{-ip(x-y)} \qquad (2,6)$$



Therefore

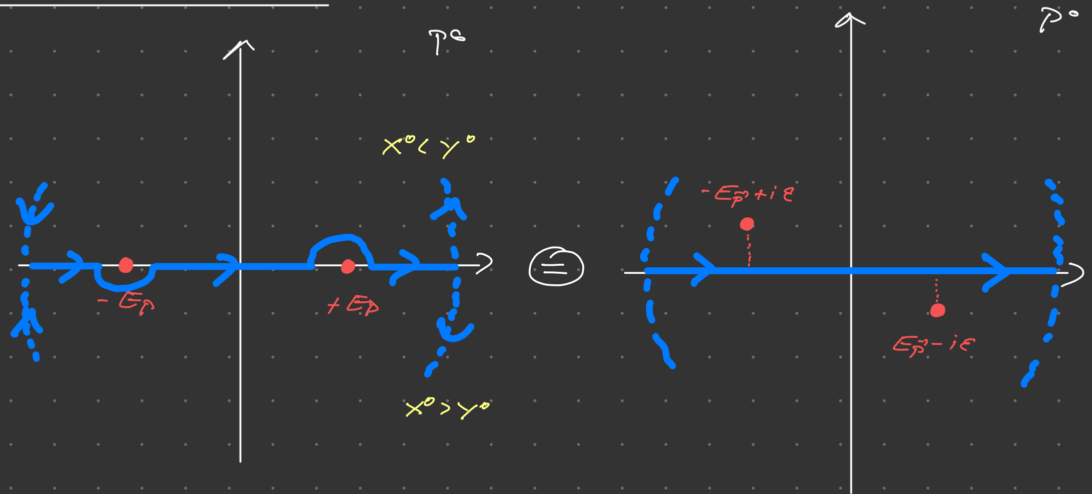
$$\boxed{D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle = (2,6) + \gamma}$$

2. Interpretation:

$$(\partial^2 + m^2)\, D_R(x-y) \overset{!}{=} -i\,\delta^{(4)}(x-y) \qquad (2,7)$$

$\rightarrow$ D_R is the <u>retarded Green's function</u> of the Klein-Gordon operator

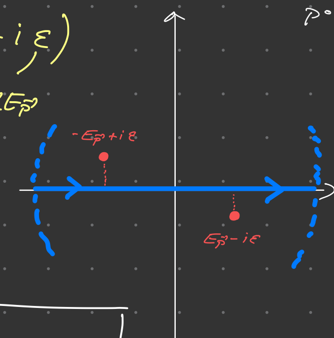
3. Alternative contour γ^*:



- $x^0 > y^0$: close contour below
- $x^0 < y^0$: close contour above

$$\boxed{D_F(x-y) \equiv (2.6) + y^* = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}} \quad \text{Feynman Propagator}$$

$$p^0 \approx \pm\left(E_p - i\frac{\tilde{\varepsilon}}{2E_{\vec{p}}}\right) = \pm(E_p - i\underbrace{\varepsilon}_{\tilde{\varepsilon}/2E_{\vec{p}}})$$



We find (using (2.6) and (2.5)):

$$\boxed{\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y) & \text{for } x^0 > y^0 \\ D(y-x) & \text{for } x^0 < y^0 \end{cases} \\ &= \theta(x^0 - y^0)\langle 0|\phi(x)\phi(y)|0\rangle \\ &\quad + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle \\ &\equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle \end{aligned}}$$

with <u>time-ordering (meta) operator</u> T

> Later: Feynman propagator
> \+
> Interactions
> $\downarrow$
> Feynman rules
> Feynman diagrams