# Aside: Poincaré and Lorentz group

- Euclidean space: $\|x\|_2^2 = x_1^2 + \dots + x_n^2 \rightarrow d(x,y)_2 = \|x-y\|$

  $\mathbb{R}^4$

  $\rightarrow \|Ux\| = \|x\| \rightarrow O(n)$, $O(3)$ $\Big\}$ Isometries of Euclidean space: $d(f(x), f(y)) = d(x,y)$

  $\rightarrow \vec{x}' = \vec{x} + \vec{a}$

  $\rightarrow \vec{x}' = U\vec{x} + \vec{a}$

- Minkowski space:

  $\mathbb{R}^{1,3} (\hat{=} \mathbb{R}^4)$

  $\rightarrow x_\mu x^\mu = g^{\mu\nu} x_\nu x_\mu$

  $\rightarrow \|\Lambda x\|_M = \|x\|_M$

  $\rightarrow \Lambda g \Lambda^T = g$

  $\Lambda^\mu{}_\nu \, g_{\mu\rho} \, \Lambda^\rho{}_\sigma = g_{\nu\sigma}$

  $\rightarrow O(1,3)$

  $\Big\}$ Isometries of Minkowski space

  $d_M(x,y) = \|x-y\|_M$

  $\rightarrow x' = \Lambda x + a$

  $\rightarrow$ Poincaré group $P(1,3)$

# Poincaré group

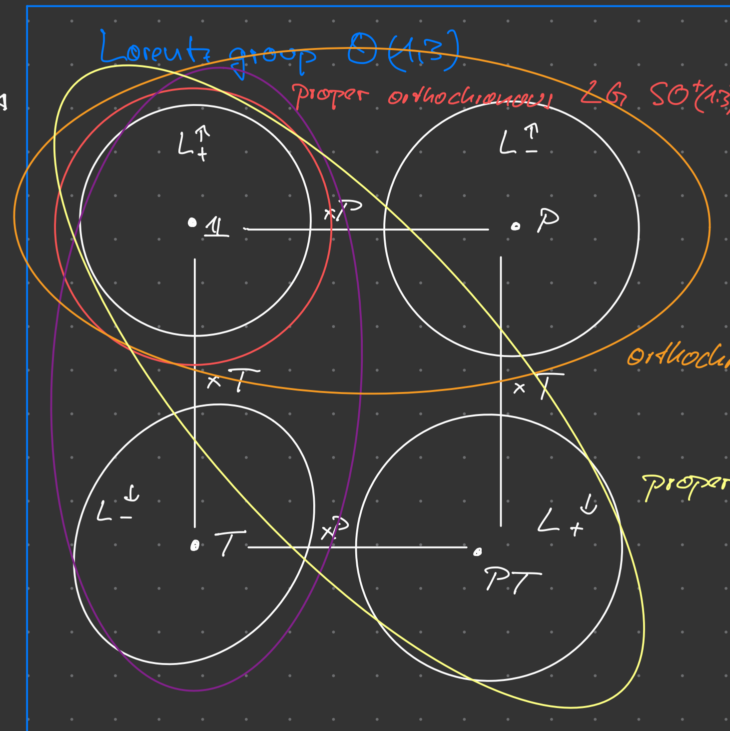
$P(1,3) = T_{1,3} \rtimes O(1,3)$

$(\Lambda, a) : \quad \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

**Translation group $T_{1,3}$**

$a : \quad \tilde{x}^\mu = x^\mu + a^\mu$

$\rtimes$




orthochronous LG

$(\Lambda, a)\cdot(\Lambda', a')$

$= (\Lambda\Lambda', a + \Lambda a')$

$\operatorname{sign}\Lambda^0{}_0 = +1$

$\uparrow$ = no time inv.

$\downarrow$ = time inv.

$\operatorname{sign}\Lambda^0{}_0 = -1$

$L^{\uparrow/\downarrow}_{\pm}$

$+ = \det\Lambda = +1$

$- = \det\Lambda = -1$

Parity

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Time reversal

# 3. The Dirac Field

## 3.1. The Dirac Equation

1. Observation: Lorentz symmetry of the KG equation:

a) Coordinate transformations: $x' = \Lambda x$ + Field transformation: $\phi'(x') = \phi(x)$

b) $\exists\, \phi$ with $(\partial^2 + m^2)\phi(x) = 0 \quad \forall x$

c) $\longrightarrow \quad \phi'(x) = \phi(\Lambda^{-1}x)$ is a new solution:

$$(g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\,\phi'(x) = \left[g^{\mu\nu}(\Lambda^{-1})^\sigma{}_\mu\,\partial_\sigma\,(\Lambda^{-1})^\rho{}_\nu\,\partial_\rho + m^2\right]\phi(\Lambda^{-1}x)$$

$$= (g^{\sigma\rho}\partial_\sigma\partial_\rho + m^2)\,\phi(\Lambda^{-1}x)$$

$$= (\partial^2 + m^2)\,\phi(\Lambda^{-1}x) \overset{\text{solution}}{=} 0$$

2. Observation: Vector fields under Rotations: $\vec{\phi}'(x) = R\,\vec{\phi}(R^{-1}x)$

$\rightarrow$ In general a field $\phi(x) \in \mathbb{C}^n$ can transform under LT as

$$\phi'_a(x) = M_{ab}(\Lambda)\,\phi_b(\Lambda^{-1}x) \qquad a = 1, \dots, n$$

where

$$M(\Lambda')M(\Lambda)\,\phi(\Lambda^{-1}\Lambda'^{-1}x) \overset{!}{=} M(\Lambda'\Lambda)\,\phi((\Lambda'\Lambda)^{-1}x)$$

is a n-dimensional representation of the LG $SO^+(1,3)$.

3. We want a first-order relativistic field equation:

$$(\partial^\mu\partial_\mu + m^2)\,\phi = 0 \qquad \Longrightarrow \qquad (i\,\square^\mu\partial_\mu + \text{const})\,\phi = 0$$

4. Then Combine 1+2)

a) Coordinate transformation: $x' = \Lambda x$ + Field transformation $\phi'(x') = M(\Lambda)\phi(x)$

b) $\exists\,\phi$ with $(i\,\square^\mu\partial_\mu + \text{const})\,\phi(x) = 0 \quad \forall x$

c) When is $\phi'(x) = M(\Lambda)\phi(\Lambda^{-1}x)$ is new solution?

$$(i\,\square^\mu\partial_\mu + \text{const})\,\phi'(x) = \left[i\,\square^\mu(\Lambda^{-1})^\nu{}_\mu\,\partial_\nu + \text{const}\right]M(\Lambda)\phi(\Lambda^{-1}x) \overset{!}{=} 0$$

$M^{-1}(\Lambda)\times|$

$$\Longleftrightarrow \left[i\,\underbrace{M^{-1}(\Lambda)\,\square^\mu M(\Lambda)(\Lambda^{-1})^\nu{}_\mu}_{\overset{!}{=}\,\square^\nu}\partial_\nu + \text{const}\right]\phi(\Lambda^{-1}x) \overset{!}{=} 0$$

$\rightarrow$ $\square^\mu \equiv \gamma^\mu$ must be $4\times 4$ matrices with

$$\boxed{M^{-1}(\Lambda)\,\gamma^\mu\, M(\Lambda) = \Lambda^\mu{}_\nu\,\gamma^\nu} \qquad (3,1)$$

5. How to find $\gamma^\mu$ and $M(\Lambda)$? $SO^+(1,3)$ is a <u>Lie group:</u>

- $\Lambda = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}\right] \overset{\omega\ll 1}{\approx} \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}$

  $(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu\right)$

- $M(\Lambda) = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right] \overset{\omega\ll}{\approx} \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta} \qquad (3,2)$

$\omega_{\alpha\beta}$ antisymmetric $4\times4$ tensor $\rightarrow$ 3 rotations (angles) + 3 boosts (rapidities)

$\rightarrow$ Infinitesimal form of (3,1):

$$[\gamma^\mu, S^{\alpha\beta}] \overset{!}{=} (\mathcal{J}^{\alpha\beta})^\mu{}_\nu\,\gamma^\nu \overset{!}{=} i\left(g^{\alpha\mu}\gamma^\beta - g^{\beta\mu}\gamma^\alpha\right) \qquad (3,3)$$

- $\mathcal{J}^{\alpha\beta} \longrightarrow$ Lie algebra of the Lorentz group:

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] \overset{!}{=} i\left(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\sigma}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho}\right) \qquad (3.4)$$

$S^{\alpha\beta}$ $\quad S^{\alpha\beta}$ $\quad S^{\alpha\beta}$ $\quad S^{\alpha\beta}$ $\quad S^{\alpha\beta}$ $\quad S^{\alpha\beta}$

**6. Solution:** Dirac's trick: $\exists\ \gamma^\mu$ such that

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}\, \mathbb{1}_{4\times 4} \qquad \text{(Dirac algebra)}}$$

Then

$$\boxed{S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]} \qquad (3.5)$$

satisfies the Lorentz algebra (3.4) and (3.3).

**7. Representations:**

- At least 4-dimensional
- All 4-dimensional representations are unitarily equivalent

- We use the <u>Weyl representation</u> (chiral representation)

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}$$

- Henceforth: $\Lambda_{\frac{1}{2}} = \mathcal{M}(\Lambda)$

8. Setting const $= -m$, we find

$$\boxed{(i\gamma^\mu \partial_\mu - m)\underbrace{\psi}_{\mathbb{R}^{1,3} \to \mathbb{C}^4} = 0 \qquad \text{(Dirac equation)}} \tag{3.6}$$

9. The components of the Dirac spinor field satisfy the KG equation:

$$\underbrace{(-i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)}_{\hat{=}\,(\partial^2 + m^2)} \psi = 0$$

## 10. Dirac adjoint

a) First try: $\psi^\dagger \psi \longrightarrow \psi'^\dagger \psi' = \psi^\dagger \underbrace{\Lambda_{\frac{1}{2}}^\dagger \Lambda_{\frac{1}{2}}}_{\neq \mathbb{1}} \psi \neq \psi^\dagger \psi$

$\longrightarrow$ $\Lambda_{\frac{1}{2}}$ is <u>not</u> unitary

b) Define

$$\boxed{\bar{\psi} = \psi^\dagger \gamma^0 \qquad \text{(Dirac adjoint)}}$$

$\leadsto \quad \bar{\psi}' \psi' = \bar{\psi} \Lambda_{\frac{1}{2}}^{-1} \Lambda_{\frac{1}{2}} \psi = \bar{\psi} \psi \longrightarrow$ Lorentz scalar

## 11. Lagrangian:

$$\boxed{\mathcal{L}_{\text{Dirac}} = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m \right) \psi}$$

$\leadsto$ Euler-Lagrange equations yield the Dirac equation

## Note 3.1

- Let $\sigma^\mu \equiv (1, \vec{\sigma})^T$ and $\bar{\sigma}^\mu \equiv (1, -\vec{\sigma})^T$ and $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$

→ Dirac equation:

$$\begin{pmatrix} -m & i\sigma\partial \\ i\bar{\sigma}\partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

$\mathbb{C}^2$

- $\psi_L$ and $\psi_R$ are called left- and right-handed <u>Weyl spinors</u>

- They do not mix under Lorentz transformations:

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \qquad \text{(Boosts, antihermitian)}$$

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \qquad \text{(Rotations, hermitian)}$$

Aside: $SO^+(1,3) \sim SU(2) \times SU(2)$

→ $(s, m)$ $\quad \left.\begin{matrix} s = 0, \frac{1}{2}, 1, \ldots \\ m = 0, \frac{1}{2}, 1, \ldots \end{matrix}\right\}$ $\Lambda_{\frac{1}{2}}: \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ $\Big\}$ $\Lambda^\mu{}_\nu: \left(\frac{1}{2}, \frac{1}{2}\right)$

- For $m=0$, the Dirac equation decouples into the <u>Weyl equations</u>:

$$i\bar{\sigma}\partial\psi_L = 0 \quad \text{and} \quad i\sigma\partial\psi_R = 0$$

## 3.2 Free-Particle Solutions of the Dirac Equation

1. (3.6) $\Rightarrow (\partial^2 + m^2)\psi = 0$, therefore

$$\psi^{\pm}(x) = \psi^{\pm}(p)\cdot e^{\mp ipx} \quad \text{with } p^2 = m^2 \text{ and } p^0 = E_p > 0 \qquad (3.7)$$

2. (3.7) in (3.6) yields:

$$(\pm\gamma^\mu p_\mu - m)\,\psi^{\pm}(p) = \begin{pmatrix} -m & \pm p\sigma \\ \pm p\bar{\sigma} & -m \end{pmatrix}\begin{pmatrix} \psi_L^{\pm} \\ \psi_R^{\pm} \end{pmatrix} = 0 \qquad (3.8)$$

3. Note: ($\Rightarrow$ Problemset 3)

- $(p\sigma)(p\bar{\sigma}) = p^2 = m^2$
- Eigenvalues of $p\sigma$ and $p\bar{\sigma}$: $p^0 \pm |\vec{p}|$ → for $m>0$ positive spectrum
  → $\sqrt{p\sigma}$, $\sqrt{p\bar{\sigma}}$ are Hermitian and invertible

4. $\measuredangle \quad \psi_L^\pm = \sqrt{p\sigma}\, \xi^\pm$ with arbitrary, normalized $\left((\xi^\pm)^\dagger(\xi^\pm)=1\right)$ spinor $\xi^\pm \in \mathbb{C}^2$

$\sqrt{p\sigma}\cdot\sqrt{p\bar\sigma} = m$

$$(3.8) \Rightarrow \quad -m\sqrt{p\sigma}\,\xi^\pm \pm p\sigma\,\psi_R^\pm = 0 \quad \Longleftrightarrow \quad \psi_R^\pm = \pm \frac{m}{\sqrt{p\sigma}}\,\xi^\pm = \pm\sqrt{p\bar\sigma}\,\xi^\pm$$

5. Solutions:

$\xi^+ \to \xi \qquad \xi^- \to \eta$

$\psi^+ \to u \qquad \psi^- \to v$

Basis states: $\xi^s$ for $\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$

(same for $\eta^s$)

$$\boxed{\underline{\psi}^+(x) = \underbrace{\begin{pmatrix}\sqrt{p\sigma}\,\xi^s\\ \sqrt{p\bar\sigma}\,\xi^s\end{pmatrix}}_{u^s(p)} e^{-ipx}} \qquad \text{(positive frequency solution)}$$

$$\boxed{\underline{\psi}^-(x) = \underbrace{\begin{pmatrix}\sqrt{p\sigma}\,\eta^s\\ -\sqrt{p\bar\sigma}\,\eta^s\end{pmatrix}}_{v^s(p)} e^{+ipx}} \qquad \text{(negative frequency solution)}$$

with $p^2 = m^2$, $p^0 > 0$, $s = 1, 2$

## 6. Some relations: ($\Rightarrow$ P-Set 3)

- Orthonormality:

Let $\bar{u}^s \equiv (u^s)^\dagger \gamma^0$, and $\bar{v}^s \equiv (v^s)^\dagger \gamma^0$, then

$*$ $\bar{u}^r u^s = 2m\delta^{rs}$ and $(u^r)^\dagger u^s = 2E_{\vec{p}}\delta^{rs}$

$*$ $\bar{v}^r v^s = -2m\delta^{rs}$ and $(v^r)^\dagger v^s = 2E_{\vec{p}}\delta^{rs}$

$*$ $\bar{v}^r u^s = \bar{u}^r v^s = 0$ $\{ (u^r)^\dagger v^s \neq 0$ and $(v^r)^\dagger u^s \neq 0$ i.g. $\}$

$*$ $(u^r)^\dagger(\vec{p})\, v^s(-\vec{p}) = (v^r)^\dagger(-\vec{p})\, u^s(\vec{p}) = 0$

- Spin sums:

Let $\not{B} \equiv \gamma^\mu B_\mu$ (Feynman slash notation), e.g. $\not{p} = \gamma^\mu p_\mu$, then

$$\sum_s u^s(p)\,\bar{u}^s(p) = \not{p} + m\cdot\mathbb{1}$$

$$\sum_s v^s(p)\,\bar{v}^s(p) = \not{p} - m\cdot\mathbb{1}$$

(3,10)