3.3. Dirac Field Bilinears

1. Definition:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \overset{\text{Weyl basis}}{=} \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

with $(\gamma^5)^\dagger = \gamma^5$, $(\gamma^5)^2 = \mathbb{1}$, $\{\gamma^5, \gamma^\mu\} = 0$ for $\mu = 0,1,2,3$

Implies $[\gamma^5, S^{\mu\nu}] = 0 \longrightarrow (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

2. The following bilinears $\bar{\Psi}\Gamma\Psi$ have definite transformation properties under the Lorentz group:

$\Gamma = 1$	scalar	$\times 1$
γ^μ	vector	$\times 4$
$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = i\gamma^{[\mu}\gamma^{\nu]}$	tensor	$\times 6$
$\gamma^\mu\gamma^5$	pseudo-vector	$\times 4$
γ^5	pseudo-scalar	$\times 1$

$SO^+(1,3)$

$P \longrightarrow -1$

pseudo vectors

$-\vec{a} \times -\vec{b} = \vec{c}$

$\vec{c} \to -\vec{c}$

$\vec{L} = \vec{r} \times \vec{p}$

For example,

$$j^{\mu'} = \bar{\psi}'\gamma^{\mu}\psi' = \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}\gamma^{\mu}\Lambda_{\frac{1}{2}}\psi = \Lambda^{\mu}{}_{\nu}\overbrace{\bar{\psi}\gamma^{\nu}\psi}^{j^{\nu}} = \Lambda^{\mu}{}_{\nu}j^{\nu}$$

$\Rightarrow$ j^{μ} is conserved Noether current corresponding to the continuous symmetry $\psi \to e^{i\alpha}\psi$ of the Dirac Lagrangian

3.4. Quantization of the Dirac Field

1. Lagrangian : $\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi = \bar{\psi}(i\not{\partial} - m)\psi$

2. Canonical momentum: $\Pi_a = \dfrac{\partial\mathcal{L}}{\partial\dot{\psi}_a} = i\psi_a^*$

3. Hamiltonian: $H = \int d^3x \; \psi^{\dagger}\underbrace{[-i\alpha\nabla + m\beta]}_{= H_D}\psi$ with $\beta = \gamma^0$, $\vec{\alpha} = \gamma^0\vec{\gamma}$

$\to$ Expand ψ in eigenmodes of H_D to diagonalize H

4. Eigenmodes: $H_D u^s(\vec{p})e^{i\vec{p}\vec{x}} = E_{\vec{p}}$ -"- and $H_D v^s(\vec{p})e^{-i\vec{p}\vec{x}} = -E_{\vec{p}}$ -"-

$[\; (i\gamma^0\partial_0 + i\vec{\gamma}\nabla - m)\psi = 0 \;]$

5. Mode expansion:

$$\psi(\vec{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a^s_{\vec{p}} u^s(\vec{p}) e^{i\vec{p}\vec{x}} + b^s_{\vec{p}} v^s(\vec{p}) e^{-i\vec{p}\vec{x}} \right] \qquad (3.11)$$

6. Use

$$H_D \psi(\vec{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left[a^s_{\vec{p}} u^s(\vec{p}) e^{i\vec{p}\vec{x}} - b^s_{\vec{p}} v^s(\vec{p}) e^{-i\vec{p}\vec{x}} \right]$$

then (orthonormality relations (3.9))

$$H = \int d^3x \, \psi^+ H_D \psi \overset{\circ}{=} \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left(a^{+s}_p a^s_p - b^{s+}_p b^s_p \right)$$

First try: Commutator

7. Canonical quantization with equal-time commutators:

$$[\psi_a(\vec{x}), \Pi_b(\vec{y})] = i\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \iff [\psi_a(\vec{x}), \psi^+_b(\vec{y})] = \delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \qquad (3.12)$$

$$[\psi_a(\vec{x}), \psi_b(\vec{y})] = 0$$

8 $\hat{=}$ Mode algebra

$$[a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}] = [b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})$$

$$[a^r_{\vec{p}}, b^{s(\dagger)}_{\vec{q}}] = 0$$

Show: (3.11) and spin sums (3.10) $\longrightarrow$ (3.12)

$$[\psi_a(\vec{x}), \psi^\dagger_b(\vec{y})] = \cancel{\psi\psi^\dagger - \psi^\dagger\psi}$$

$\rightarrow$ irreducible Representation = bosonic Fock space

9. Problem: $(b^{s\dagger}_{\vec{p}})^n |0\rangle$ has energy $-nE_{\vec{p}} \xrightarrow{n\to\infty} -\infty$

$\rightarrow$ no stable vacuum state

10. Fix (?): $b \longleftrightarrow b^\dagger$

a) $\psi(\vec{x}) = \ldots [a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} \ldots]$

b) $H = \ldots (a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^s_{\vec{p}} b^{s\dagger}_{\vec{p}})$

c) $[b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})$

d) $H = \ldots (a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}}) + \text{const}$

e) $[H, b^{s\dagger}_p] = E_{\vec{p}} b^\dagger_{\vec{p}} \longrightarrow b^{s\dagger}_{\vec{p}}$ creates a particle with positive energy! $\longrightarrow H \geq 0$

f) <u>But:</u> $\| b_{\vec{p}}^{s\dagger}|0\rangle \|^2 = \langle 0|[b_{\vec{p}}^{s}, b_{\vec{p}}^{s\dagger}]|0\rangle = -(2\pi)^3\delta^{(3)}(0) < 0$

$\longrightarrow$ negative norm states

11. <u>Conclusion: (3,12) implies</u>

- either an instability of the vacuum
- or a loss of unitarity

$\longrightarrow$ no consistent quantisation possible!

Second try: Anticommutators

7. Canonical quantisation with equal-time <u>anti</u>commutators:

$$\boxed{\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y}) \quad \text{and} \quad \{\psi_a(\vec{x}), \psi_b(\vec{y})\} = 0}$$

8. $\overset{\circ}{\longrightarrow}$ Mode algebra:

$$\boxed{\{a_{\vec{p}}^r, a_{\vec{q}}^{s\dagger}\} = \{b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger}\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q}) \quad \text{and} \quad \{a_{\vec{p}}^r, b_{\vec{q}}^{s(\dagger)}\} = 0 \qquad (3,13)}$$

$\longrightarrow$ irreducible Representation = <u>fermionic</u> Fock space

9. <u>Problem:</u> $b^{s\dagger}_{\vec{p}}|0\rangle$ has energy $-E_{\vec{p}}$ & infinite sum over momenta

$\longrightarrow$ still no stable vacuum state

10. Fix (??): $b \leftrightarrow b^\dagger$

a) <u>Hamiltonian:</u>

$$\boxed{\begin{aligned} H &= \sum_s \int \frac{d^3p}{(2\pi)^3} E_p \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^s_{\vec{p}} b^{s\dagger}_{\vec{p}} \right) \\ &= \sum_s \int \frac{d^3p}{(2\pi)^3} E_p \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right) - \cancel{\infty} \end{aligned}}$$

b) The mode algebra (3,13) is <u>invariant</u> under $b \leftrightarrow b^\dagger$!

$\longrightarrow$ Unitarity is preserved <u>and</u> Hamiltonian is bounded from below

11. Heisenberg picture:

With $\quad e^{iHt} a^s_{\vec{p}} e^{-iHt} \overset{\circ}{=} a^s_{\vec{p}} e^{-iE_{\vec{p}}t}$ and $\quad e^{iHt} b^s_{\vec{p}} e^{-iHt} \overset{\circ}{=} b^s_{\vec{p}} e^{-iE_{\vec{p}}t}$

and $\quad \psi(x) = e^{iHt} \psi(\vec{x}) e^{-iHt}$ we find

$$\boxed{\begin{aligned} \psi(x) &= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a^s_{\vec{p}} u^s(p) e^{-ipx} + b^{s\dagger}_{\vec{p}} v^s(p) e^{ipx} \right] \\ \bar{\psi}(x) &= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a^{s\dagger}_{p} \bar{u}^s(p) e^{ipx} + b^s_{\vec{p}} \bar{v}^s(p) e^{-ipx} \right] \end{aligned}}$$

Continuous symmetries & Conserved charges

- Time translation $\longrightarrow$ Hamiltonian
- Spatial translations $\longrightarrow$ momentum operator

$$\vec{P} \overset{\circ}{=} \int d^3x \; \psi^\dagger(-i\nabla)\psi \overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3} \vec{p} \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$$

- Spatial rotations $\longrightarrow$ angular momentum operator $\vec{J}$

$$\vec{J} \hat{=} \int d^3x \; \psi^\dagger \left(\vec{x} \times (-i\nabla) + \frac{1}{2} \vec{\Sigma} \right) \psi \quad \text{with} \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \qquad (3.14)$$

- Global phase rotations $e^{i\alpha}\psi \xrightarrow{\circ}$ conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi \longrightarrow$ conserved charge:

$$\boxed{\begin{aligned} Q = \int d^3x : \psi^\dagger \psi : &\hat{=} \sum_s \int \frac{d^3p}{(2\pi)^3} :\left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^s_{-\vec{p}} b^{s\dagger}_{-\vec{p}} \right): \\ &= \sum_s \int \frac{d^3p}{(2\pi)^3} \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right) + \cancel{\infty} \end{aligned}}$$

Excitations = Particles

$a^{s\dagger}_{\vec{p}}|0\rangle$: Fermion with Energy $E_{\vec{p}}$

$b^{s\dagger}_{\vec{p}}|0\rangle$: Antifermion with Energy $E_{\vec{p}}$

Momentum $\vec{p}$

momentum $\vec{p}$

spin $J = \frac{1}{2}$ (polarization s)

spin $J = \frac{1}{2}$ (polarization opposite to s)

and charge $Q = +1$

and charge $Q = -1$

Note 3.2:

- The two states for $s=1,2$ suggests a spin-$\frac{1}{2}$ representation
- To show this, the action of $\vec{J}$ (see (3,14)) on one-particle states must be studied
- One finds for particles at rest:

$$J_z\, a^{s\dagger}_{\vec{0}}|0\rangle = \pm\frac{1}{2}\, a^{s\dagger}_{\vec{0}}|0\rangle \quad \text{and} \quad J_z\, b^{s\dagger}_{\vec{0}}|0\rangle = \mp\frac{1}{2}\, b^{s\dagger}_{\vec{0}}|0\rangle$$

$$+\ \xi^{s=1}=\begin{pmatrix}1\\0\end{pmatrix},\quad -\ \xi^{s=2}=\begin{pmatrix}0\\1\end{pmatrix}$$

Lorentz transformations:

1. $\exists$ Lorentz transformation $\Lambda \in SO^+(1,3)$ on single particle state $|\vec{p},s\rangle \equiv \sqrt{2E_{\vec{p}}}\, a^{s\dagger}_{\vec{p}}|0\rangle$

$$|\vec{p},s\rangle \longmapsto U(\Lambda)\,|\vec{p},s\rangle$$

$U(\Lambda)$: representation of $SO^+(1,3)$ on Fock space

2. Special case: quantization axis parallel to boost and/or rotation axis 

→ spin polarizations do not mix:

$$U(\Lambda)\, a^s_{\vec p}\, U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec p}}{E_{\vec p}}}\; a^s_{\Lambda\vec p}$$

(in general: mixing of $a^1_{\vec p} \leftrightarrow a^2_{\vec p}$)

3. Then:

$$\langle \vec p, s | \vec q, r\rangle = \underbrace{2E_{\vec p}(2\pi)^3\, \delta^{(3)}(\vec p - \vec q)\, \delta^{rs}}_{\text{Lorentz invariant}} = \langle \vec p, s | \underbrace{U^\dagger(\Lambda)\, U(\Lambda)}_{=\mathbb{1}} | \vec q, r\rangle$$

→ $U(\Lambda)$ is unitary

4. Now we have 3 representations:

Λ	acts on 4-vectors in $\mathbb{R}^{1,3}$	$D=4$	not unitary
$\Lambda_{\frac{1}{2}}$	acts on bispinors in $\mathbb{C}^2 \oplus \mathbb{C}^2$	$D=4$	not unitary
$U(\Lambda)$	acts on states in fermionic Fock space	$D=\infty$	unitary

5. Action by conjugation on field operators:

$$\overset{\circ}{\longrightarrow} \quad \boxed{U(\Lambda)\,\Psi(x)\,U^{-1}(x) = \Lambda_{\frac{1}{2}}^{-1}\,\Psi(\Lambda x)}$$