Spin-Statistics Theorem

- Observation:

Klein-Gordon field ϕ: Spin 0 (scalar) $\rightarrow$ commutator $\rightarrow$ bosonic excitations

Dirac field ψ: Spin $\frac{1}{2}$ (spinor) $\rightarrow$ anticommutator $\rightarrow$ fermionic excitations

- Spin-Statistics theorem:

$$\left.\begin{array}{r}\text{Lorentz invariance}\\ \text{Causality}\\ \text{Positive energies}\\ \text{Positive norms}\end{array}\right\} \Longrightarrow \boxed{\begin{array}{l}\text{Integer spin} \Leftrightarrow \text{Bosons}\\ \text{Half-integer spin} \Leftrightarrow \text{Fermions}\end{array}}$$

- The proof is elaborate and technical

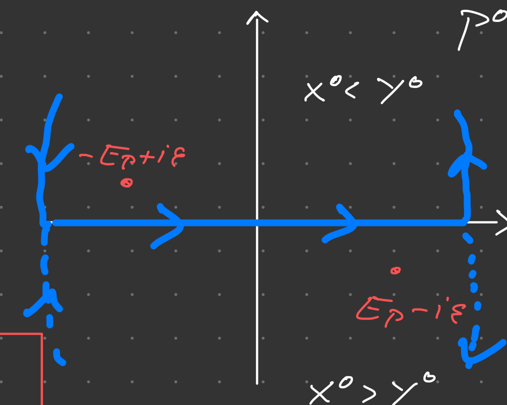
Dirac Propagator

1. Propagation amplitude:

$$\langle 0|\, \psi_a(x)\, \bar{\psi}_b(y)\, |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}\, e^{-ip(x-y)} \underbrace{\sum_s u_a^s(p)\, \bar{u}_b^s(p)}_{(\not{p}+m)_{ab}}$$

$\bar{\psi}_b(y)\, \psi_a(x)$ $\qquad$ $\overbrace{v_a^s(p)\, \bar{v}_b^s(p)}^{(\not{p}-m)_{ab}}$

$$\overset{x^0>y^0}{=} \; \underset{x^0<y^0}{-} \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\varepsilon}\, e^{-ip(x-y)}$$



2. Therefore we define the Feynman propagator:

$$\boxed{\begin{aligned} S_F^{ab}(x-y) &= \int \frac{d^4p}{(2\pi)^3} \frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\varepsilon}\, e^{-ip(x-y)} \\ &= \begin{cases} \langle 0|\, \psi_a(x)\, \bar{\psi}_b(y)\, |0\rangle & \text{for } x^0>y^0 \\ -\langle 0|\, \bar{\psi}_b(y)\, \psi_a(x)\, |0\rangle & \text{for } x^0<y^0 \end{cases} \\ &\equiv \langle 0|\, T\, \psi_a(x)\, \bar{\psi}_b(y)\, |0\rangle \end{aligned}}$$

Note: For $t_1>t_2$ it is

$$T\, \psi(t_2)\, \psi(t_1) \equiv -\psi(t_1)\, \psi(t_2)$$

for fermion fields!

3. Retarded Green's function:

$$S_R^{ab}(x-y) \equiv \theta(x^0-y^0)\langle 0|\{\psi_a(x), \bar\psi_b(y)\}|0\rangle \overset{\circ}{=} (i\not{\partial}_x + m)_{ab}\, D_R(x-y)$$

<u>Causality:</u>

- Measurable operators: $\hat{O}(x) = \sum \prod_{i=1}^{\text{even } N} \{\psi_i(x) \vee \partial\psi_i(x) \vee \partial^2\psi_i\}$

  Example: $j^\mu = \bar\psi\gamma^\mu\psi$ (but <u>not</u>: $\psi_a + \psi_a^+$!)

- $\rightarrow$ Causality $\Longleftrightarrow$ $\{\psi_a(x), \bar\psi_b(y)\} = 0$ for $(x-y)^2 < 0$

  $y = (t, \vec{y})$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-ip(x-y)}$$

We find:

$$\{\psi_a(x), \bar\psi_b(y)\} \overset{\circ}{=} (i\not{\partial}_x + m)_{ab}\left[D(x-y) - D(\underbrace{y-x}_{x-y})\right]$$

$$\overset{(x-y)^2<0}{=} (i\not{\partial}_x + m)\underbrace{\left[D(x-y) - D(x-y)\right]}_{=0} = 0$$

3.5 Discrete Symmetries of the Dirac Theory

Review of the Lorentz group

- Lorentz group $O(1,3)$ = Lie group with four disconnected components
- Continuous Lorentz transformations = Proper orthochronous Lorentz group $SO^+(1,3)$
- Four components connected by discrete transformations:

$$\boxed{\begin{array}{ll} \text{Parity:} & (t,\vec{x}) \longmapsto (t,-\vec{x}) \\ \text{Time reversal:} & (t,\vec{x}) \to (-t,\vec{x}) \end{array}}$$

Parity

(⇒ Problemset 4)

(Note: $U(P) = P$)

- Unitary representation on the Hilbert space:

$$U(P)\, a^s_{\vec{p}}\, U^{-1}(P) = \overbrace{\eta_a}^{+1}\, a^s_{-\vec{p}} \qquad \text{and} \qquad U(P)\, b^s_{\vec{p}}\, U^{-1}(P) = \overbrace{\eta_b}^{-1}\, b^s_{-\vec{p}}$$

$(\vec{L} = \vec{r} \times \vec{p})$

- Equivalen to

$$\boxed{U(\mathcal{P})\psi(t,\vec{x})U^{-1}(\mathcal{P}) = \underbrace{\gamma^0}_{P_{\frac{1}{2}}}\psi(t,-\vec{x}) \quad (\leftrightarrow \Lambda_{\frac{1}{2}})}$$

- Dirac field bilinears:

$$U(\mathcal{P})\bar{\psi}\psi U^{-1}(\mathcal{P}) \overset{\circ}{=} \bar{\psi}\psi(t,-\vec{x}) \quad \longrightarrow \text{ scalar}$$

$$U(\mathcal{P})\bar{\psi}\gamma^5\psi U^{-1}(\mathcal{P}) \overset{\circ}{=} -\bar{\psi}\gamma^5\psi(t,-\vec{x}) \quad \rightarrow \text{ pseudo-scalar}$$

Time Reversal

1. Time reversal should ...

(Note: $U(T) \overset{\text{often}}{=} T$)

- $U(T)\psi(t,\vec{x})U^{-1}(T) = T_{\frac{1}{2}}\psi(-t,\vec{x})$
- $U(T)a^s_{\vec{p}}U^{-1}(T) = a^{?}_{-\vec{p}}$
- flip spin ($\vec{L} = \vec{r}\times\vec{p} \mapsto -\vec{L}$)
- be a symmetry of the Dirac theory ($[U(T), H] = 0$)
- obey $U^{-1}(T) = U^{+}(T)$ ($\Leftarrow$ Wigner's theorem)

2. Problem:

$$\psi(t,x) = e^{iHt}\,\psi(\vec{x})\,e^{-iHt}$$

$U(T)$ antilinear

$$\Rightarrow \quad U(T)\psi(t,x)U^{-1}(T) = e^{-iHt}\,U(T)\,\psi(\vec{x})\,U^{-1}(T)\,e^{\pm iHt}$$

$$\Rightarrow \quad T_{\frac{1}{2}}\psi(-t,\vec{x})\,|0\rangle = e^{-iHt}\,T_{\frac{1}{2}}\,\psi(\vec{x})\,|0\rangle$$

$$\Rightarrow \quad T_{\frac{1}{2}}\,e^{-iHt}\,\psi(\vec{x})\,|0\rangle = e^{-iHt}\,T_{\frac{1}{2}}\,\psi(\vec{x})\,|0\rangle$$

$$\Rightarrow \quad \underbrace{e^{-2iHt}}_{\text{time dependent! / time independent!}}\overset{0\cdot t}{} T_{\frac{1}{2}}\psi(\vec{x})\,|0\rangle \overset{(\not{\,})}{=} \underbrace{T_{\frac{1}{2}}\,\psi(\vec{x})\,|0\rangle}_{\text{time independent!}} \longrightarrow \text{not possible!}$$

3. Solution: $U(T)$ must be antiunitary (antilinear):

$$\boxed{U(T)\,c = c^{*}\,U(T) \qquad \text{for } c \in \mathbb{C}} \tag{3.15}$$

Note:

$$\underset{\text{antiunitary}}{U(T)} = \underset{\text{unitary matrix}}{U_T} \circ \overset{\text{complex conjugation}}{K}$$

4. Transformation of spin:

a) Spinors: 2 spin basis ξ^s $(s=1,2)$ along arbitrary axis $\vec{u}$:

$$\xi^1 = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \quad \text{and} \quad \xi^2 = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

$$= |\uparrow\rangle \qquad\qquad = |\downarrow\rangle$$

Note:

$T = -i\sigma^y \circ K$

"Time reversed" (= flipped) spinors:

$$\overline{\xi^s} \equiv -i\sigma^2(\xi^s)^* \implies \begin{Bmatrix} \overline{\xi^1} \\ \overline{\xi^2} \end{Bmatrix} = \begin{Bmatrix} \xi^2 \\ -\xi^1 \end{Bmatrix} \qquad (3.16)$$

$$\vec{u}\cdot\vec{\sigma}\,\xi = +\xi \implies \vec{u}\cdot\vec{\sigma}(-i\sigma^2\xi^*) = -i\sigma^2(-\vec{u}\vec{\sigma})^*\xi^* = i\sigma^2\xi^* = -(-i\sigma^2\xi^*)$$

b) Bispinors:

$$u^s(p) = \begin{pmatrix} \sqrt{p\sigma}\,\xi^s \\ \sqrt{p\bar{\sigma}}\,\xi^s \end{pmatrix} \quad \text{and} \quad v^s(p) = \begin{pmatrix} \sqrt{p\sigma}\,\overline{\xi^s} \\ -\sqrt{p\bar{\sigma}}\,\overline{\xi^s} \end{pmatrix} \qquad (3.17)$$

$$\overline{u^s}(p) \equiv \begin{pmatrix} \sqrt{p\sigma}\,\overline{\xi^s} \\ \sqrt{p\bar{\sigma}}\,\overline{\xi^s} \end{pmatrix} \quad \text{and} \quad \overline{v^s}(p) = \begin{pmatrix} \sqrt{p\sigma}\,\overline{\overline{\xi^s}} \\ -\sqrt{p\bar{\sigma}}\,\overline{\overline{\xi^s}} \end{pmatrix}$$

c) Define the mods:

$$\begin{Bmatrix} \overline{a^1_{\vec{p}}} \\ \overline{a^2_{\vec{p}}} \end{Bmatrix} \equiv \begin{Bmatrix} a^2_{\vec{p}} \\ -a^1_{\vec{p}} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \overline{b^1_{\vec{p}}} \\ \overline{b^2_{\vec{p}}} \end{Bmatrix} \equiv \begin{Bmatrix} b^2_{\vec{p}} \\ -b^1_{\vec{p}} \end{Bmatrix} \qquad (3.18)$$

d) Let $\tilde{p} \equiv (p^0, -\vec{p})$ and show:

$$\overline{u^s(\tilde{p})} \stackrel{\circ}{=} -\gamma^1\gamma^3 \left[u^s(p)\right]^* \qquad (3.19)$$

$$\overline{v^s(\tilde{p})} \stackrel{\circ}{=} -\gamma^1\gamma^3 \left[v^s(p)\right]^*$$

Notes:

- $\overline{\overline{\xi^s}} = -\xi^s$ used in $\overline{v^s}$
- Use (3.17) and $\sqrt{\tilde{p}\sigma}\,\sigma^2 = \sigma^2 \sqrt{p\sigma^*}$

$$\begin{pmatrix} -i\sigma^2 & \\ & -i\sigma^2 \end{pmatrix} \sim \begin{pmatrix} \sigma^1\sigma^3 & \\ & \sigma^1\sigma^3 \end{pmatrix} = \begin{pmatrix} & \sigma^1 \\ -\sigma^1 & \end{pmatrix}\begin{pmatrix} & \sigma^3 \\ -\sigma^3 & \end{pmatrix}$$

5. Definition:

Antilinearity (3.15)

$$\left.\begin{aligned} U(T)\, a^s_{\vec{p}}\, U^{-1}(T) &\equiv \overline{a^s_{-\vec{p}}} \\ U(T)\, b^s_{\vec{p}}\, U^{-1}(T) &\equiv \overline{b^s_{-\vec{p}}} \end{aligned}\right\} \Rightarrow U(T)\,\psi(t,\vec{x})\,U^{-1}(T) \stackrel{\circ}{=} \underbrace{\gamma^1\gamma^3}_{T_{\frac{1}{2}}}\,\psi(-t,\vec{x})$$

(Use (3.19) and (3.18) and $\overline{a^2_p}\,\overline{u^2(p)} = a^1_p u^1(p)$)

6. Dirac field bilinears (example: $j^\mu = \overline{\psi}\gamma^\mu\psi$):

$$U(T)\, j^\mu(t,x)\, U^{-1}(T) \stackrel{\circ}{=} \begin{cases} +\, j^\mu(-t,x) & \mu = 0 \\ -\, j^\mu(-t,\vec{x}) & \mu = 1,2,3 \end{cases}$$

Charge Conjugation

1. Discrete, non-spacetime symmetry that exchanges particles and antiparticles:

$$\boxed{U(C)\, a^s_{\vec{p}}\, U^{-1}(C) = b^s_{\vec{p}} \quad \text{and} \quad U(C)\, b^s_{\vec{p}}\, U^{-1}(C) = a^s_{\vec{p}}}$$

Note: $U(C) \stackrel{\text{often}}{=} C$

2. Use (3.17) to show:

$$u^s(p) \stackrel{\circ}{=} -i\gamma^2 (v^s(p))^* \quad \text{and} \quad v^s(p) \stackrel{\circ}{=} -i\gamma^2 (u^s(p))^*$$

3. Then

$$U(C)\,\psi(x)\,U^{-1}(C) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[-i\gamma^2 (v^s(p))^* \, b^s_{\vec{p}}\, e^{-ipx} - i\gamma^2 (u^s(p))^* \, a^{s\dagger}_{p}\, e^{ipx} \right]$$

$$= -i\gamma^2 (\psi^\dagger)^T = -i\,(\overline{\psi}\gamma^0\gamma^2)^T$$

4. Therefore:

$$\boxed{\begin{aligned} &U(C)\,\Psi\,U^{-1}(C) = -i\left(\overline{\Psi}\gamma^0\gamma^2\right)^T \\ \text{and}\quad &U(C)\,\overline{\Psi}\,U^{-1}(C) \overset{!}{=} -i\left(\gamma^0\gamma^2\Psi\right)^T \end{aligned}}$$

$$= \underbrace{-i\gamma^2}_{C_{\frac{1}{2}}}\Psi^*$$

Note:

Recall γ^0 and γ^2 are symmetric! $(\gamma^2)^T = \gamma^2$

5. Dirac field bilinears:

$$U(C)\,\overline{\Psi}\Psi\,U^{-1}(C) \overset{!}{=} \overline{\Psi}\Psi \qquad (\text{Scalar})$$

$$U(C)\,\overline{\Psi}\gamma^\mu\Psi\,U^{-1}(C) \overset{!}{=} -\overline{\Psi}\gamma^\mu\Psi \qquad (\text{Vector})$$

$$H = \sum_s \int d\tilde{p}^3\, E_p\left(a_p^{s\dagger}a_p^s + b_p^{s\dagger}b_p^s\right)$$

Note 3.3

- Any relativistic QFT must be invariant under $SO^+(1,3)$
- The (classical) Dirac equation $(i\gamma^\mu\partial_\mu - m)\Psi = 0$ is $\{C, P, T\}$ - invariant
- The (quantized) Dirac theory is $\{C, P, T\}$ - invariant: $[H, U(X)] = 0$ for $X = P, T, C$

- Weak interactions (of standard model) violate C and P but preserve CP and T

(⇒ Wu experiment)

- Rare processes (decay of neutral kaons) violate CP and T but preserve CPT
- CPT seems to be a perfect symmetry of nature
- <u>CPT theorem:</u>

$$\left.\begin{array}{r} SO^+(1,3) \text{ invariance} \\ \text{Causality} \\ \text{Locality} \\ \text{Stable vacuum} \end{array}\right\} \Longrightarrow \text{CPT symmetry}$$