4. Interacting Fields and Feynman Diagrams

4.1. Preliminaries

- In the following: $$H_{int} = \int d^3x\, \mathcal{H}_{int}(\phi(x)) = -\int d^3x\, \mathcal{L}_{int}(\phi(x))$$

  $\mathcal{L}_{int}$ only a function ϕ

- Examples:

1. ϕ^4-theory:

$$\boxed{\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4(x)}$$

λ: dimensionless coupling constant

→ EOMs:

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$$

→ Cannot be solved by Fourier modes!

2. Yukawa theory:

$$\boxed{\mathcal{L}_{Yukawa} = \underbrace{\bar{\Psi}(i\partial\!\!\!/ - m)\Psi}_{\text{Dirac}} + \underbrace{\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2}_{\text{Klein-Gordon}} - \underbrace{g\bar{\Psi}\Psi\phi}_{\text{Interaction}}}$$

g: dimensionless coupling constant

3. Quantum Electrodynamics (QED):

$$\boxed{\begin{aligned}\mathcal{L}_{QED} &= \underbrace{\bar{\psi}(i\not{D}-m)\psi}_{\text{Dirac}} - \underbrace{\tfrac{1}{2}(F_{\mu\nu})^2}_{\text{Maxwell}} - e\,\bar{\psi}\gamma^{\mu}\psi\, A_{\mu} \\ &= \bar{\psi}(i\not{D}-m)\psi - \tfrac{1}{4}(F_{\mu\nu})^2\end{aligned}}$$

$e = -|e| < 0$: Electron charge

$D_{\mu} \equiv \partial_{\mu} + ieA_{\mu}(x)$: Gauge covariant derivative

(Replace $\partial \mapsto D$: "minimal coupling")

U(1) Gauge symmetry:

$$A'_{\mu} = A_{\mu} - \frac{1}{e}\partial_{\mu}\alpha(x)$$

$$\psi' = e^{i\alpha(x)}\psi(x)$$

→ Equation of motion:

$$(i\not{D}-m)\psi(x) = 0 \quad \text{and} \quad \partial_{\mu}F^{\mu\nu} = e\,j^{\nu} \quad (j^{\nu} = \bar{\psi}\gamma^{\nu}\psi)$$

- No known exactly solvable interacting QFTs in $D > 1+1$!

  $\rightarrow$ Perturbation theory

In $D = 1+1$ there are examples of exactly solvable QFTs:

Conformal Field Theories (CFT)

4.2. Perturbation Expansion of Correlation Functions

Problemset 5

1. Goal: Two-point Green's function $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$ of ϕ^4-theory

   $|0\rangle$: Ground state of <u>free</u> theory

   $|\Omega\rangle$: Ground state of <u>interacting</u> theory

2. Remember: $\langle 0 | T \phi(x) \phi(y) | 0 \rangle = D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon}$

3. Now: $H_{\phi^4} = \underbrace{H_0}_{\text{KG Hamiltonian}} + \underbrace{\int d^3x \, \frac{\lambda}{4!} \phi^4(\vec{x})}_{H_{int}: \text{ Interaction } = \text{ Perturbation}}$

4. Todo:

$$\phi(x) = e^{iHt}\phi(\vec{x})e^{-iHt}$$

Express $\begin{Bmatrix} \phi(x) \\ |\Omega\rangle \end{Bmatrix}$ in terms $\begin{Bmatrix} \text{free field } \phi_I(x) \\ \text{free vacuum } |0\rangle \end{Bmatrix}$

5. ∄ Reference time t_0, then

$$\phi(t_0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{i\vec{p}\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}}\right)$$

6. Definitions:

$$\phi(t,x) \equiv e^{iH(t-t_0)}\phi(t_0,\vec{x})e^{-iH(t-t_0)} \qquad \text{(Heisenberg picture)}$$

$$\phi_I(t,x) \equiv e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)} \qquad \text{(Interaction picture)}$$

Then:

$$\phi_I(t,\vec{x}) \overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{-ipx} + a^\dagger_{\vec{p}}e^{ipx}\right)$$

and $\phi(t,\vec{x}) = U^\dagger(t,t_0)\,\phi_I(t,\vec{x})\,U(t,t_0)$ with $U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$ (4.1)

7. The time-evolution operator is determined by $U(t_0,t_0)=\mathbb{1}$ and the differential equation

$$i\partial_t U(t,t_0) \stackrel{\circ}{=} H_I(t)\,U(t,t_0) \tag{4.2}$$

with

$$H_I(t) = e^{iH_0(t-t_0)}\,H_{int}\,e^{-iH_0(t-t_0)} = \int d^3x\,\frac{\lambda}{4!}\,\phi_I^4(t,\vec{x}) \tag{4.3}$$

8. The solution of (4.2) is given by the Dyson series:

$$\boxed{\begin{aligned} U(t,t_0) &= \mathbb{1} + (-i)\int_{t_0}^{t} dt_1\, H_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1\, dt_2\, T\{H_I(t_1)\,H_I(t_2)\} + \ldots\ldots \\ &\equiv T\exp\left\{-i\int_{t_0}^{t} ds\, H_I(s)\right\} \qquad (4.4)\end{aligned}}$$

(⊕ Problemset 5)

9. Properties:

$$\left.\begin{aligned} &\bullet\ U(t,t') = e^{iH_0(t-t_0)}\,e^{-iH(t-t')}\,e^{-iH_0(t'-t_0)} \\ &\bullet\ U^{-1}(t,t') = U^{\dagger}(t,t') \\ &\bullet\ U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3) \qquad (t_1 \geq t_2 \geq t_3) \end{aligned}\right\} \quad (4.5)$$

10. Ground state $|\Omega\rangle$?

$\lambda \ll 1 \to \langle\Omega|0\rangle \neq 0$

$$\to \quad e^{-iHT}|0\rangle = \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

Then (since $E_n > E_0$ for $n \neq 0$)

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

$$\overset{\circ}{=} \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(t_0+T)}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)\,|0\rangle \qquad (4.6)$$

$\Rightarrow$ Problemset 5

Similar: $$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|\,U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \qquad (4.7)$$

11. Two-point correlator: (let $x^0 > y^0 > t_0$)

$$\langle\Omega|\,\phi(x)\,\phi(y)\,|\Omega\rangle = (4.7)\times(4.1)\times(4.1)\times(4.6)$$

$\phi(y)\,\phi(x)$

$$\overset{(4.5)}{=} \lim_{T\to\infty(1-i\varepsilon)} \mathcal{N}_T^{-1}\,\langle 0|\,U(T,\overset{y^0}{x^0})\,\phi_I(\overset{y}{x})\,\overset{y^0\;x^0}{U(x^0,y^0)}\,\phi_I(\overset{x}{y})\,U(\overset{x^0}{y^0},-T)\,|0\rangle$$

with $\mathcal{N}_T \underset{\langle\Omega|\Omega\rangle=1}{\overset{(4.7)\times(4.6)}{=}} \langle 0|\,U(T,t_0)\,U(t_0,-T)\,|0\rangle \overset{(4.5)}{=} \langle 0|\,U(T,-T)\,|0\rangle$

For $x^0 \gtrless y^0$ arbitrary $\longrightarrow$

$$\boxed{\langle\Omega|\,T\phi(x)\phi(y)\,|\Omega\rangle \overset{(4.4)}{=} \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\,T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]\right\}|0\rangle}{\langle 0|\,T\left\{\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]\right\}|0\rangle} \qquad (4.8)}$$

(4.3) and (4.8) $\longrightarrow$ expand time-ordered exponential in orders of λ:

$$\langle\Omega|\,T\phi(x)\phi(y)\,|\Omega\rangle = \sum \dots \underbrace{\langle 0|\,T\,\phi_I(x_1)\,\phi_I(x_2)\dots\phi_I(x_n)\,|0\rangle}_{\text{How to evaluate this efficiently?}} \dots$$

4.3. Wick's Theorem

1. Define

$$\phi_I(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\, a_p\, e^{-ipx}}_{\equiv\, \phi_I^+(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\, a^\dagger_{\vec p}\, e^{+ipx}}_{\equiv\, \phi_I^-(x)}$$

Useful because

$\phi_I^+|0\rangle = 0$

$\langle 0|\phi_I^- = 0$

2. Observation $x^0 > y^0$ and $n=2$:

$x^0 < y^0$

$$T\,\phi_I(x)\phi_I(y) = \phi_I^+\underset{y}{(x)}\,\phi_I^+\underset{x}{(y)} + \phi_I^+\underset{y}{(x)}\,\phi_I^-\underset{x}{(y)} + \phi_I^-\underset{y}{(x)}\,\phi_I^+\underset{x}{(y)} + \phi_I^-\underset{y}{(x)}\,\phi_I^-\underset{x}{(y)}$$

$$= \phi_I^+\underset{y}{(x)}\,\phi_I^+\underset{x}{(y)} + \phi_I^-\underset{x}{(y)}\,\phi_I^+\underset{y}{(x)} + \phi_I^-\underset{y}{(x)}\,\phi_I^+\underset{x}{(y)} + \phi_I^-\underset{y}{(x)}\,\phi_I^-\underset{x}{(y)}$$

$$+ \left[\phi_I^+\underset{y}{(x)},\, \phi_I^-\underset{x}{(y)}\right]$$

3. Definitions:

Contraction:

$$\overbrace{\phi(x)\phi(y)} \equiv \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0 \end{cases} \;\hat{=}\; D_F(x-y)\cdot \mathbb{1}$$

Normal order:

$$:a_1^{(\dagger)} \cdots\cdots a_n^{(\dagger)}: \;\equiv\; (\text{creation operators}) \times (\text{annihilation operators})$$

Example: $:a^\dagger a: \, = a^\dagger a$, $\quad :a a^\dagger: \, = a^\dagger a$, $\quad :\phi^+(x)\phi^-(y): \, = \phi^-(y)\phi^+(x)$

Beware: $a^\dagger a = \, :a a^\dagger: \, = \, :a^\dagger a + 1: \, = \, :a^\dagger a: + :1: \, = a^\dagger a + 1$

$$\longrightarrow \quad T\,\phi(x)\phi(y) = \, :\phi(x)\phi(y): \, + \, \overbracket{\phi(x)\phi(y)}$$

$$\longrightarrow \quad \langle 0|\, T\,\phi(x)\phi(x)\, |0\rangle = \overbrace{\langle 0|\, :\phi(x)\phi(y):\, |0\rangle}^{=0} + \langle 0|\, \overbrace{\overbracket{\phi(x)\phi(y)}}^{D_F(x-y)\cdot \mathbb{1}}\, |0\rangle$$

$$= D_F(x-y)$$

$$:e^{i\phi(x)}:$$

$$:\sum_n \frac{(i\phi(x))^n}{n!}:$$

$$= \sum_n^{!} \frac{:i\phi(x)^n:}{n!}$$

4. Wick's theorem: (Proof $\Rightarrow$ Problemset 5)

$$\boxed{\begin{array}{l} T\{\phi(x_1)\dots\phi(x_n)\} = \, :\phi(x_1)\dots\phi(x_n) + \text{all possible contractions}: \\[1em] \text{where} \quad :A\overbracket{\phi_i B \phi_j} C: \, \equiv D_F(x_i - x_j)\cdot :ABC: \end{array}}$$

Example: $T(\phi_1\phi_2\phi_3\phi_4) = \, :\phi_1\phi_2\phi_3\phi_4: + :\overbracket{\phi_1\phi_2}\phi_3\phi_4: + :\overbracket{\phi_1\phi_2\phi_3}\phi_4: + \dots\dots$

$$+ :\overbracket{\phi_1\phi_2}\,\overbracket{\phi_3\phi_4}: + :\phi_1\phi_2\phi_3\phi_4: + \dots\dots$$

5. Corollary:

$$\boxed{\langle 0 | T\{\phi(x_1) \dots \phi(x_n)\} | 0\rangle = \text{all } \underline{\text{full}} \text{ contractions}}$$