Solid State Theory, Exercises X

Prof. Hans Peter Büchler SS 2012, 11th of July 2012

The Bogoliubov-Valatin-Transformation

We consider the following Hamiltonian :

$$\mathcal{H} - \mu\mathcal{N} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} n_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left[\Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + \Delta^{*}_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - \Delta_{\mathbf{k}} b^{*}_{\mathbf{k}} \right]. \quad (1)$$

This Hamiltonian can be diagonalized by a suitable linear transformation to define new Fermi operators $\alpha_{\mathbf{k}}$, the Bogoliubov-Valatin-Transformation :

$$\begin{aligned} c_{\mathbf{k}\uparrow} &= u^{*}_{\mathbf{k}} \alpha_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \alpha^{\dagger}_{-\mathbf{k}\downarrow} \\ c^{\dagger}_{-\mathbf{k}\downarrow} &= -v^{*}_{\mathbf{k}} \alpha_{\mathbf{k}\uparrow} + u_{\mathbf{k}} \alpha^{\dagger}_{-\mathbf{k}\downarrow} \\ \alpha_{\mathbf{k}\uparrow} &= u_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} \\ \alpha_{-\mathbf{k}\downarrow} &= v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} + u_{\mathbf{k}} c_{-\mathbf{k}\downarrow}, \end{aligned}$$

with $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$.

a) We first study this Hamiltonian for non-interacting fermions with $\Delta = 0$. Demonstrate that the new operators can be interpreted has hole creation operator for $\mathbf{k} < \mathbf{k}_f$ and particle creation operator for $\mathbf{k} > \mathbf{k}_f$.

b) We now study the superconducting situation with $\Delta \neq 0$. Calculate explicitly the commutation rules for the new operators $\alpha_{\mathbf{k}\sigma}$, $\{\alpha^{\dagger}_{\mathbf{k}\uparrow}, \alpha_{\mathbf{k}\uparrow}\}$ and $\{\alpha_{\mathbf{k}\uparrow}, \alpha_{-\mathbf{k}\downarrow}\}$, to get conditions on $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$.

c) Rewrite the Hamiltonian in terms of the Bogoliubov operators $\alpha_{\mathbf{k}\sigma}$.

d) To diagonalize this Hamiltonian we choose u_k, v_k such that the non diagonal terms vanishes. For this, regroup the terms proportional to $\alpha_{-\mathbf{k}\downarrow}\alpha_{\mathbf{k}\uparrow}$ and $\alpha^{\dagger}_{\mathbf{k}\downarrow}\alpha^{\dagger}_{-\mathbf{k}\downarrow}$ in a system of two equations and show that it gives us the condition :

$$\frac{\Delta^{*}_{\mathbf{k}} v_{\mathbf{k}}}{u_{\mathbf{k}}} = -\xi_{\mathbf{k}} \pm \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2} \equiv E_{\mathbf{k}} - \xi_{\mathbf{k}}, \quad (2)$$

and give the resulting condition on $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ in function of $\xi_{\mathbf{k}}$ and $E_{\mathbf{k}}$.

e) Using the previous results show that the Hamiltonian has the following form :

$$\mathcal{H} - \mu\mathcal{N} = \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}} + \Delta_{\mathbf{k}} b^{*}_{\mathbf{k}}) + \sum_{\mathbf{k}} E_{\mathbf{k}} (\alpha^{\dagger}_{\mathbf{k}\uparrow}\alpha_{\mathbf{k}\uparrow} + \alpha^{\dagger}_{-\mathbf{k}\downarrow}\alpha_{-\mathbf{k}\downarrow}). \quad (3)$$

Solutions due on : 20th of July, 2012