Solid State Theory, Exercises VII

Prof. Hans Peter Büchler SS 2012, 15 of June 2012

1. **Thomas-Fermi Screening**
   The susceptibility $\chi_{sc}(\mathbf{q})$ is defined in terms of the Fourier transforms of the charge density $\rho_{ind}(\mathbf{r})$ and the total potential $\phi(\mathbf{r})$ (including internal and external sources) by

$$\rho_{ind}(\mathbf{r}) = \chi_{sc}(\mathbf{q})\phi(\mathbf{r}). \tag{1}$$

   In the Thomas-Fermi theory the induced charge density of the equilibrium distribution of particles in an external potential is taken to be

$$\rho_{ind}(\mathbf{r}) = -e\{n_0[\mu + e\phi(\mathbf{r})] - n_0(\mu)\}, \tag{2}$$

   where μ is the chemical potential in the absence of an electric potential.

   (a) Under the assumption that $\phi(\mathbf{r})$ is small, write $\rho_{ind}(\mathbf{r})$ as a derivative and find from this an expression for $\chi_{sc}(\mathbf{q})$. What is $\chi_{sc}(\mathbf{q})$ for an electron gas?

   (b) The static dielectric function is defined through $\epsilon(\mathbf{q}) = 1 - (4\pi/q^2)\chi_{sc}(\mathbf{q})$ for all $\mathbf{q}$. Write $\epsilon(\mathbf{q})$ for the electron gas and from this find an expression for the total potential $\phi(\mathbf{q})$ in the presence of an extra Coulomb charge, $\phi_{Coulomb}(\mathbf{q}) = 4\pi e/q^2$. What is the Thomas-Fermi screening length q_{TF}^{-1} for the electron gas? Transform $\phi(\mathbf{q})$ back into real space and give a short interpretation of q_{TF}^{-1}.

2. **Landau-Silin Theory**
   The Landau-Silin transport equation in the presence of an electric field is

$$(-\omega + \mathbf{q}\cdot\mathbf{v_p})\delta n_{\mathbf{p}} + [\mathbf{q}\cdot\mathbf{v_p}(\sum_{\mathbf{p}'} f_{\mathbf{p},\mathbf{p}'}\delta n_{\mathbf{p}'}) + ie\mathbf{E}\cdot\mathbf{v_p}]\delta(\epsilon_{\mathbf{p}} - \mu) = 0 \tag{3}$$

   (a) Using the simple model $f_{\mathbf{p},\mathbf{p}'} = f_0 \equiv (\pi^2\hbar^2/m^* k_{\mathbf{F}})F_0$, find an expression for the density $\langle\rho(\mathbf{q},\omega)\rangle = \sum_{\mathbf{p}} \delta n_{\mathbf{p}}$ induced in response to a longitudinal field $\mathbf{E}(\mathbf{q},\omega)||\mathbf{q}$. Your answer should be written in terms of the functions $F(\lambda) = 1 - (\lambda/2)log[(\lambda + i\eta + 1)/(\lambda + i\eta - 1)]$ where $\lambda = \omega/v_F q$.
   The longitudinal dielectric constant is

$$\epsilon(\mathbf{q},\omega) = 1 + \frac{4\pi ie\langle\rho(\mathbf{q},\omega)\rangle}{\mathbf{q}\cdot\mathbf{E}(\mathbf{q},\omega)}. \tag{4}$$

   (b) Use your answer for $\langle\rho(\mathbf{q},\omega)\rangle$ to show that for the simple model,

$$\epsilon(\mathbf{q},\omega) = 1 + \frac{3}{v_F^2 q^2}\frac{\omega_p^2 F(\lambda)}{1 + F_0 F(\lambda)}. \tag{5}$$

   (c) Discuss the form of $Im\,\epsilon$ and $Re\,\epsilon$ as a function of ω/qv_F for the different regimes of F_0. In the case of strong coupling ($F_0 \gg 1$) show that the zero sound pole in $\chi(\mathbf{q},\omega) = (q^2/4\pi)[\epsilon(\mathbf{q},\omega) - 1]$ occurs at large frequencies, $\lambda \approx \sqrt{F_0/3}$.(Hint : expand

$F(\lambda)$ in powers of $1/\lambda$). Show that at large frequencies the dielectric constant may be written,

$$\epsilon(\mathbf{q},\omega) = 1 - \frac{\omega_p^2}{(\omega + i\eta)^2 - s^2 q^2}, \tag{6}$$

where the sound velocity is $s = v_F\sqrt{F_0/3}$.

(d) What will happen in the response to an external field when the frequency and wavelength of the probe satisfy $\omega = sq$?

3. **Landau Damping of Plasmons**
You will now calculate the critical wavevector q_c above which a dispersive plasmon mode is Landau-damped. The dispersion of the plasmons is

$$\omega_{\mathbf{q}} = \omega_{\mathbf{p}}[1 + \frac{3}{10}(\frac{qv_F^0}{\omega_{\mathbf{p}}})^2 + \cdots]. \tag{7}$$

The upper limit of the single-pair excitations is given by

$$\omega = qv_F^0 + \frac{1}{2}\frac{q^2}{m}, \tag{8}$$

so that the imaginary part of the dielectric constant, $\epsilon''(\mathbf{q},\omega) = 0$ if $\omega > qv_F^0 + \frac{1}{2}\frac{q^2}{m}$. Writing $q_c = \alpha\omega_{\mathbf{p}}/v_F^0$ sketch the variation of the coefficient α as a function of the ratio $\omega_{\mathbf{p}}/\epsilon_F$. Indicate on your sketch the region corresponding to typical values of the Fermi energy and the plasma frequency in a metal, and give an order of magnitude estimate for q_c.

Solutions due on : 22nd of June, 2012