Solid State Theory, Exercises IX

Prof. Hans Peter Büchler SS 2012, 4th of July 2012

1. The Cooper problem

   Consider a pair of electrons in a singlet state, described by the symmetric spatial wave function
$$\phi(\mathbf{r}-\mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3}\chi(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{1}$$
   In the momentum representation the Schrödinger equation has the form
$$(E - 2\frac{\hbar^2 k^2}{2m})\chi(\mathbf{k}) = \int \frac{d\mathbf{k}'}{(2\pi)^3}V(\mathbf{k},\mathbf{k}')\chi(\mathbf{k}'). \tag{2}$$
   We assume that the two electrons interact in the presence of a degenerate free electron gas, whose existence is felt only via the exclusion principle, i.e., electron levels with $k < k_F$ are forbidden to each of the two electrons, which gives the constraint :
$$\chi(\mathbf{k}) = 0,\ k < k_F. \tag{3}$$
   We take the interaction of the pair to have the simple attractive form
$$\begin{aligned} V(\mathbf{k_1},\mathbf{k_2}) &\equiv -V,\ \epsilon_F \le \frac{\hbar^2 k_i^2}{2m} \le \epsilon_F + \hbar\omega,\ i = 1,2; \\ &\equiv 0,\ \text{otherwise,} \end{aligned}$$
   and look for a bound-state solution to the Schrödinger equation (Eq. 2) consistent with the constraint (Eq. 3). Since we are considering only one-electron levels which in the absence of the attraction have energies in excess of $2\epsilon_F$, a bound state will be one with energy E less than $2\epsilon_F$, and the binding energy will be
$$\Delta = 2\epsilon_F - E. \tag{4}$$

   a) From the Schödinger equation (Eq. 2), derive the equation for the bound state energy E
$$1 = V\int_{\epsilon_F}^{\epsilon_F+\hbar\omega} d\epsilon \frac{N(\epsilon)}{2\epsilon - E}, \tag{5}$$
   where $N(\epsilon)$ is the density of state and $\epsilon = \frac{\hbar^2 k^2}{2m}$.

   b) Show that the above equation has a solution with $E < 2\epsilon_F$ for arbitrarily weak V, provided that $N(\epsilon \neq 0)$. Note the crucial role played by the exclusion principle : If the lower cutoff were not ϵ_F, but 0, then since $N(0) = 0$, there would not be a solution for arbitrarily weak coupling.

   c) Assuming that $N(\epsilon)$ differs negligibly from $N(\epsilon_F)$ in the range $\epsilon_F < \epsilon < \epsilon_F + \hbar\omega$, show that the binding energy is given by
$$\Delta = 2\hbar\omega \frac{e^{-2/N(\epsilon_F)V}}{1 - e^{-2/N(\epsilon_F)V}}, \tag{6}$$

or, in the weak-coupling limit $(N(\epsilon_F)V \ll 1)$

$$\Delta = 2\hbar\omega e^{-2/N(\epsilon_F)V}. \tag{7}$$

2. Josephson effects

   Two superconductors $i = 1, 2$ occupy the negative and positive half space, with the wavefunction given by $\psi_i = \sqrt{\rho_i} e^{i\phi_i}$. The Schrödinger equations are given by

$$i\hbar \frac{\partial}{\partial t}\psi_1 = E_1\psi_1 + K\psi_2$$
$$i\hbar \frac{\partial}{\partial t}\psi_2 = K\psi_1 + E_2\psi_2,$$

   where K is the coupling between the two superconductors.

   Solve the Schrödinger equation with the given wavefunctions and derive the solution

$$\frac{d\phi_1}{dt} - \frac{d\phi_2}{dt} = \frac{E_2 - E_1}{\hbar}.$$

   What happens if a voltage $eU = E_2 - E_1$ is applied ? Compute the current

$$I_{1\to 2} = e\frac{\partial}{\partial t}|\psi_1|^2 = e(\psi_1^* \frac{\partial \psi_1}{\partial t} + \frac{\partial \psi_1^*}{\partial t}\psi_1) \tag{8}$$

Solutions due on : 13th of July, 2012