Exercise 1 - Momentum distribution function (3 points)

The momentum distribution function is given by

$$n_\sigma(\mathbf{k}) = \langle \psi_0 | f^\dagger_{\mathbf{k},\sigma} f_{\mathbf{k},\sigma} | \psi_0 \rangle, \tag{1}$$

(a) Prove the relation of it with the Green's function

$$n_\sigma(\mathbf{k}) = \frac{1}{2\pi i} \int_C d\omega \, \left[ G_c(\mathbf{k},\omega) + G_{inc}(\mathbf{k},\omega) \right] \tag{2}$$

where the contour of integration $C$ is shown in Fig. 4.9 of the lecture notes.

(b) Consider non-interacting fermions where the Green's function only contains the coherent part and is explicitly given in Eq. 4.338 of the lecture notes. Calculate the corresponding momentum distribution function. Also consider the interacting case, where the quasiparticle weight $z(\mathbf{k}) < 1$ and the Green's function has a finite incoherent part with no poles. Qualitatively discuss the behavior of $n_\sigma(\mathbf{k})$.

Exercise 2 - Ground state energy (3 points)

(a) Show that the expectation value of the total kinetic energy in the ground state of a many-body fermion system is given by

$$\langle \hat{T} \rangle = -i \int d^3\mathbf{x} \lim_{\mathbf{x}' \to \mathbf{x}} \left[ \frac{-\hbar^2 \nabla^2}{2m} \right] \sum_\sigma G_{\sigma,\sigma}(\mathbf{x}, t; \mathbf{x}', t^+) \tag{3}$$

where the kinetic energy operator is given by $\hat{T} = -\frac{\hbar^2 \nabla^2}{2m}$.

(b) Show that the total ground state energy is given by

$$\begin{aligned} E &= \langle \hat{H} \rangle = \langle \hat{T} + \hat{V} \rangle = \langle \hat{T} \rangle + \langle \hat{V} \rangle \\ &= -\frac{i}{2} \int d^3\mathbf{x} \lim_{\mathbf{x}' \to \mathbf{x}} \left[ i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2 \nabla^2}{2m} \right] \sum_\sigma G_{\sigma,\sigma}(\mathbf{x}, t; \mathbf{x}', t^+) \end{aligned} \tag{4}$$

Hint: To evaluate

$$\langle \hat{V} \rangle = \frac{\langle \Psi_0 | \frac{1}{2} \int d^3\mathbf{x} \int d^3\mathbf{x}' \hat{\Psi}^\dagger_\alpha(\mathbf{x}) \hat{\Psi}^\dagger_\beta(\mathbf{x}') V(|\mathbf{x} - \mathbf{x}'|) \hat{\Psi}_\beta(\mathbf{x}') \hat{\Psi}_\alpha(\mathbf{x}) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \tag{5}$$

use the Heisenberg equation of motion for $\hat{\Psi}_{H\alpha}(\mathbf{x}, t)$ to show that

$$\left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2m} \right] \hat{\Psi}_{H\alpha}(\mathbf{x}, t) = \int d^3\mathbf{x}' \hat{\Psi}^\dagger_{H\beta}(\mathbf{x}', t) V(|\mathbf{x} - \mathbf{x}'|) \hat{\Psi}_{H\beta}(\mathbf{x}', t) \hat{\Psi}_{H\alpha}(\mathbf{x}, t) \tag{6}$$

Solutions due on: 22 July, 2011