Exercise 1 - The strong coupling limit of the Hubbard model (6 points)

Consider the Hubbard model with nearest-neighbour hopping and repulsive on-site interaction

$$H = -t \sum_{\substack{<i,j> \\ \sigma}} \left(c^\dagger_{i\sigma} c_{j\sigma} + \text{h.c.} \right) + U \sum_i n_{i\uparrow} n_{i\downarrow} \ , \quad U > 0 \tag{1}$$

(a) First we consider only the interaction part of H. What are its Eigenstates? What is the groundstate and how high is its degenaracy?

(b) The Hilbert space $\mathcal{H}$ of the full Hamiltonian decomposes into a sum of subspaces, characterized by the number of doubly occupied sites

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_N \tag{2}$$

We consider P_n, the projectors into the respective subspaces and their complement $Q_n = 1 - P_n$. Show that, given an eigenstate $|\psi\rangle$ with $H|\psi\rangle = E|\psi\rangle$, and any projector P_n, the state $|\varphi\rangle = P_n|\psi\rangle$ satisfies

$$\hat{H}_n(E)|\varphi\rangle = E|\varphi\rangle \tag{3}$$

where

$$\hat{H}_n(E) = P_n H \big(1 + (E - Q_n H)^{-1} Q_n H\big) P_n \tag{4}$$

(c) Show that for a Hamiltonian of the form $H = H_A + \lambda H_B$, where

$$H_A = \sum_n E_n P_n \tag{5}$$

equation (4) can be reformulated as

$$\hat{H}_n(E) = \left(E_n + P_n H_B \sum_{k=0}^{\infty} \lambda^{k+1} \left[\sum_{\substack{m \\ m \neq n}} \frac{P_m H_B}{E - E_m} \right]^k \right) P_n \tag{6}$$

(d) Finally, we consider the strong coupling limit, eg. $U >> t$. We rescale the Hamiltonian $H \to H/U$ so that the hopping term acts as a small perturbation $\lambda \cdot H_B$ to the interaction part H_A. Since the lifting of degeneracy is small compared to the level spacing between the groundstate and the first excited state of H, we only consider the case where $n = 0$. Show that up to first order in $\lambda = 1/U$, we obtain the effective eigenvalue problem

$$\left[P_0 H_B P_0 - \frac{1}{U} \sum_{\substack{m \\ m \neq 0}} \frac{P_0 H_B P_m H_B P_0}{m} \right] |\varphi\rangle = E|\varphi\rangle \tag{7}$$

(e) Show that for the case of half-filling, the effective Hamiltonian simplifies to the well known Heisenberg model

$$H_{\text{eff}} = \frac{4t^2}{U} \sum_{<i,j>} \vec{S}_i \cdot \vec{S}_j - \frac{1}{4} \ , \qquad \text{where } \vec{S}_i = \frac{1}{2} c^\dagger_i \vec{\sigma} c_i \tag{8}$$

Solutions due on: 29 July, 2011