Exercise 1 - van Hove singularities in one dimension

(2 points)

On the last exercise sheet you calculated the dispersion relation for the one-dimensional chain,

\[ \omega(q) = \omega_0 | \sin(qa/2) | , \]

where \( \omega_0 \) is the maximum frequency for \( q \) at the zone boundary. Show that the density of states (DOS) for this case is

\[ N(\omega) = \frac{2}{\pi a \sqrt{\omega_0^2 - \omega^2}} , \]

and discuss the van Hove singularities.

Exercise 2 - Creation and annihilation operators

(2 points)

Consider the operators \( \hat{b}_{\nu q} \) and \( \hat{b}_{\nu q}^\dagger \) satisfying the canonical commutation relations for bosons

\[ [\hat{b}_{\mu q}, \hat{b}_{\nu q}^\dagger] = \delta_{\mu\nu} \delta_{q,q'} , \quad [\hat{b}_{\mu q}, \hat{b}_{\nu q}] = [\hat{b}_{\mu q}^\dagger, \hat{b}_{\nu q}^\dagger] = 0 . \]

The number operator, defined through \( \hat{n}_{\nu q} = \hat{b}_{\nu q}^\dagger \hat{b}_{\nu q} \), has eigenstates \( |n_{\nu q}\rangle \) with the property

\[ \hat{n}_{\nu q} |n_{\nu q}\rangle = n_{\nu q} |n_{\nu q}\rangle , \]

with \( n \) some positive integer.

- Show that \( \hat{b}_{\nu q} |n_{\nu q}\rangle \) and \( \hat{b}_{\nu q}^\dagger |n_{\nu q}\rangle \) are also eigenstates of \( \hat{n}_{\nu q} \) with the properties

\[ \hat{b}_{\nu q} |n_{\nu q}\rangle = \sqrt{n_{\nu q}} |n_{\nu q} - 1\rangle , \]
\[ \hat{b}_{\nu q}^\dagger |n_{\nu q}\rangle = \sqrt{n_{\nu q} + 1} |n_{\nu q} + 1\rangle . \]

- Show that there is an eigenstate \( |0\rangle \) of \( \hat{n}_{\nu q} \) with eigenvalue 0, so that

\[ \hat{b}_{\nu q} |0\rangle = 0 . \]

The state \( |0\rangle \) in general is called the vacuum.

Solutions due on: 20 May, 2011