Exercise 1 - Wave functions (2 points)

Write down the wave functions for the following systems:

(a) Three spinless bosons: two occupying the ground state and one occupying the first excited state.

(b) Three spinless fermions occupying the lowest three states of a quantum system.

Exercise 2 - The ideal Bose gas (3 points)

Consider a system of non-interacting Bosons in a three dimensional box with periodic boundary conditions. The Hamiltonian in second quantization reads

$$H = \sum_{\vec{k}} \epsilon_{\vec{k}} \, b^{\dagger}_{\vec{k}} b_{\vec{k}} \ , \quad \text{where} \ \ \epsilon_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m} \ . \tag{1}$$

(a) Calculate the density of states

(b) You have seen in the lecture that the partition function for the grand canonical ensemble in case of an ideal Bose gas reads

$$Z := \mathrm{tr}\Big[\mathrm{e}^{-\beta(H-\mu N)}\Big] = \prod_{\vec{k}} \frac{1}{1-\exp\left[\beta(\mu-\epsilon_{\vec{k}})\right]} \ . \tag{2}$$

Show that the grand potential $\Omega := -k_B T \ln Z$ can be written as

$$\Omega(T,V,\mu) = -\frac{2}{3}\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^{\infty} d\epsilon \ \frac{\epsilon^{\frac{3}{2}}}{\exp\left[\beta(\epsilon-\mu)\right]-1} \ . \tag{3}$$

(c) Show that the equation of states for the ideal Bose gas is given by

$$pV = \frac{2}{3}E \tag{4}$$

*Remarks:*

(1) For homogenous systems we have $\Omega = -pV$.

(2) Bosons respect Bose-Einstein statistics: the mean occupation number is given by

$$n(\vec{k}) = \frac{1}{\exp\left[\beta(\epsilon_{\vec{k}}-\mu)\right]-1} \ . \tag{5}$$

Which values of the chemical potential are physically reasonable?

Exercise 3 - Bose-Einstein condensation (4 points)

We examine again the ideal Bose gas from the last exercise.

(a) Consider the high temperature limit where $k_B T >> 1$ and derive the chemical potential as a function of temperature by considering the mean particle number, given through

$$N = -\frac{\partial \Omega}{\partial \mu}\Big|_{T,V} \; . \tag{6}$$

Discuss the behaviour of $\mu(T)$.

(b) Now we drop the limit of high-$T$. Show that there is nevertheless a finite temperature

$$T_0 \approx 3.31 \frac{\hbar^2}{m k_B} \left(\frac{N}{V}\right)^{\frac{2}{3}} \; , \tag{7}$$

at which the chemical potential vanishes. Verify that the occupation of the lowest energy state diverges in this case.

(c) Show that for $T \leq T_0$ the specific heat $c_V = \frac{\partial E}{\partial T}\big|_V$ is given through

$$c_V = \frac{\zeta(5/2)\Gamma(5/2)}{\zeta(3/2)\Gamma(3/2)} \cdot \frac{5}{2} N k_B \left(\frac{T}{T_0}\right)^{\frac{3}{2}} \approx 0.77 \cdot \frac{5}{2} N k_B \left(\frac{T}{T_0}\right)^{\frac{3}{2}} \; . \tag{8}$$

(d) For temperatures slightly above $T_0$ we approximate the internal energy by

$$E(T, V, \mu) \;\approx\; E(T \leq T_0, V) + \frac{3}{2} N \mu(T) \; , \tag{9}$$

and the chemical potential by

$$\mu(T) \;\approx\; -0.54\, k_B T_0 \left( \left(\frac{T}{T_0}\right)^{\frac{3}{2}} - 1 \right)^2 \; . \tag{10}$$

Calculate the specific heat in this regime.

*Hint:* You might encounter the following integral:

$$\int_0^\infty dx \, \frac{x^{a-1}}{\mathrm{e}^x - 1} = \Gamma(a)\zeta(a) \qquad \text{for } a > 1 \; . \tag{11}$$

The $\Gamma$-function and the Riemann $\zeta$-function are well tabulated in the literature.

Solutions due on: 10 June, 2011