Exercise 1 - Second quantization (3 points)

Consider a many-body system with the Hamiltonian

$$H = \sum_{k,l} a_k^\dagger \langle k|T|l\rangle a_l + \frac{1}{2}\sum_{k,l,m,n} a_k^\dagger a_l^\dagger \langle k,l|V|m,n\rangle a_m a_n, \tag{1}$$

where $a^\dagger$ and a are the creation and annihilation operators respectively.

1. Using the commutation relations for bosons, show that $[n_j, b_i] = -\delta_{i,j} b_j$ and $[n_j, b_i^\dagger] = \delta_{i,j} b_j^\dagger$, where the particle density operator n_j is defined as $n_j = b_j^\dagger b_j$.
2. Using the commutation relations for fermions, show that $[n_j, f_i] = -\delta_{i,j} f_j$ and $[n_j, f_i^\dagger] = \delta_{i,j} f_j^\dagger$, where the particle density operator n_j is defined as $n_j = f_j^\dagger f_j$.

The particle number operator is defined as $N = \sum_k a_k^\dagger a_k$

3. Show that the particle number operator N commutes with the Hamiltonian H, in case the particles are bosons.
4. Show that the particle number operator N commutes with the Hamiltonian H, in case the particles are fermions.

Exercise 2 - Density of states for the 1D and 2D Fermi gas (2 points)

You already calculated the DOS for the Fermi gas in three dimensions. Consider now the Fermi gas in one and two dimensions and derive the density of states for these cases.

Exercise 3 - Low temperature properties of the Fermi gas (2 points)

From thermodynamics, we know the pressure P can be calculated from the thermodynamic potential Ω as $P = -\Omega/V$. Based on the thermodynamic potential $\Omega(T, V, \mu)$ of the Fermi gas at low temperature, i.e., Eq. 4.247 in the lecture notes, derive the pressure in terms of E_F and T up to $O(T^2)$. Observe that even at $T = 0$, there is residue pressure in the Fermi gas, where does this pressure come from?

Solutions due on: 8 July, 2011