Exercise 1 - Second quantization  

Consider a many-body system with the Hamiltonian

\[ H = \sum_{k,l} a_{k}^\dagger \langle k|T|l \rangle a_{l} + \frac{1}{2} \sum_{k,l,m,n} a_{k}^\dagger a_{l}^\dagger \langle k,l|V|m,n \rangle a_{m} a_{n}, \tag{1} \]

where \( a^\dagger \) and \( a \) are the creation and annihilation operators respectively.

1. Using the commutation relations for bosons, show that \([n_{j}, b_{i}] = -\delta_{i,j} b_{j} \) and \([n_{j}, b_{i}^\dagger] = \delta_{i,j} b_{j}^\dagger \)

2. Using the commutation relations for fermions, show that \([n_{j}, f_{i}] = -\delta_{i,j} f_{j} \) and \([n_{j}, f_{i}^\dagger] = \delta_{i,j} f_{j}^\dagger \)

The particle number operator is defined as \( N = \sum_{k} a_{k}^\dagger a_{k} \)

3. Show that the particle number operator \( N \) commutes with the Hamiltonian \( H \), in case the particles are bosons.

4. Show that the particle number operator \( N \) commutes with the Hamiltonian \( H \), in case the particles are fermions.

Exercise 2 - Density of states for the 1D and 2D Fermi gas  

You already calculated the DOS for the Fermi gas in three dimensions. Consider now the Fermi gas in one and two dimensions and derive the density of states for these cases.

Exercise 3 - Low temperature properties of the Fermi gas  

From thermodynamics, we know the pressure \( P \) can be calculated from the thermodynamic potential \( \Omega \) as \( P = -\Omega/V \). Based on the thermodynamic potential \( \Omega(T,V,\mu) \) of the Fermi gas at low temperature, i.e., Eq. 4.247 in the lecture notes, derive the pressure in terms of \( E_{F} \) and \( T \) up to \( O(T^2) \). Observe that even at \( T = 0 \), there is residue pressure in the Fermi gas, where does this pressure come from?

Solutions due on: 8 July, 2011