Theoretische Physik III: Klassische Elektrodynamik, Exercise 1

Prof. Dr. Hans Peter Büchler, SS 2015, 15. April 2015

Information regarding the lecture and the online version of the exercise sheets can be found on the webpage `http://www.itp3.uni-stuttgart.de/`. The exercise problems are split into two types: **written** and **oral**. The **written** problems have to be turned in during the exercise groups, will be corrected by the individual tutor and handed out in the following exercise session. The **oral** exercise problems have to be prepared for the exercise session. The standard procedure is, that at the beginning of the session the prepared problems can be checked on the corresponding list. Afterwards one student is chosen to present the checked problem on the blackboard. The requirement for obtaining the exercise certificate is: turn in the **written** problems and achieve **80%** of the available points, prepare and check **66%** of the **oral** problems, and present a problem on the blackboard **twice** during the exercise period.

As a repetition, you should familiarize yourself with basic math, algebra and calculus from the lecture "Mathematische Methoden der Physik".

1. Repetition: Vector Calculus (Written) [3 pt.]

Here we recall some standard identities of vector calculus, which we will use throughout the lecture.

Definitions and conventions: We write the vectorial differentiation operators grad, div, rot using the vector $\boldsymbol{\nabla}$ of partial derivatives $\nabla_i := \partial/\partial x_i$ as

$$\operatorname{grad} F := \boldsymbol{\nabla} F\,, \qquad \operatorname{div} \mathbf{A} := \boldsymbol{\nabla}\cdot\mathbf{A}\,, \qquad \operatorname{rot} \mathbf{A} := \boldsymbol{\nabla}\times\mathbf{A}\,. \tag{1}$$

The components of a three-dimensional vector product $\mathbf{a}\times\mathbf{b}$ are given by

$$(\mathbf{a}\times\mathbf{b})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk}\, a_j b_k\,, \tag{2}$$

here ε_{ijk} is the totally anti-symmetric tensor in $\mathbb{R}^3$ with $\varepsilon_{123} = +1$.

a) Show that

$$\sum_{i=1}^{3} \varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \qquad \text{and} \qquad \frac{1}{2}\sum_{i,j=1}^{3} \varepsilon_{ijk}\,\varepsilon_{ijl} = \delta_{kl}\,. \tag{3}$$

b) Now show the following identities for vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$:

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})\,,\\
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\,,\\
(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\,.
\end{aligned} \tag{4a}$$

c) Prove the following identities for the scalar fields F and vector fields $\mathbf{A}$, $\mathbf{B}$:

$$
\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} F) &= 0\,, \\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} \times \mathbf{A}) &= 0\,, \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \Delta\mathbf{A}\,, \\
\boldsymbol{\nabla}\cdot(F\mathbf{A}) &= (\boldsymbol{\nabla} F)\cdot\mathbf{A} + F\boldsymbol{\nabla}\cdot\mathbf{A}\,, \\
\boldsymbol{\nabla} \times (F\mathbf{A}) &= (\boldsymbol{\nabla} F) \times \mathbf{A} + F\boldsymbol{\nabla} \times \mathbf{A}\,, \\
\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A})\,, \\
\boldsymbol{\nabla}\cdot(\mathbf{A} \times \mathbf{B}) &= \mathbf{B}\cdot(\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla} \times \mathbf{B})\,. \qquad (5a)
\end{aligned}
$$

2. Gauß's Theorem (Oral)

Consider the following vector fields $\mathbf{A}_i$ in two dimensions

$$
\begin{aligned}
\mathbf{A}_1 &= \Big(3xy(y-x), x^2(3y-x)\Big)\,, \\
\mathbf{A}_2 &= \Big(x^2(3y-x), 3xy(x-y)\Big)\,, \\
\mathbf{A}_3 &= \Big(x/(x^2+y^2), y/(x^2+y^2)\Big) = \mathbf{x}\big/|\mathbf{x}|^2\,. \qquad (6a)
\end{aligned}
$$

a) Compute the flux of $\mathbf{A}_i$ through the boundary of the square Q with corners $\mathbf{x} = (\pm 1, \pm' 1)$

$$I_i = \oint_{\partial Q} dx\, \mathbf{n}\cdot\mathbf{A}_i\,. \qquad (7)$$

b) Calculate the divergence of $\mathbf{A}_i$ and its integral over the area of this square Q

$$I_i' = \int_Q d^2x\, \boldsymbol{\nabla}\cdot\mathbf{A}_i\,. \qquad (8)$$

3. Stokes' Theorem (Oral)

Consider the vector field

$$\mathbf{A} = \Big(x^2 y, x^3 + 2xy^2, xyz\Big)\,. \qquad (9)$$

a) Compute the integral along the circle S around the origin in the xy-plane with radius R

$$I = \oint_S d\mathbf{x}\cdot\mathbf{A}\,. \qquad (10)$$

b) Calculate the rotation $\mathbf{B}$ of the vector field $\mathbf{A}$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}\,. \qquad (11)$$

c) Determine the flux of the rotation $\mathbf{B}$ through the disk S whose boundary is S, $\partial D = S$

$$I' = \int_D d^2x\, \mathbf{n}\cdot\mathbf{B}\,. \qquad (12)$$